LAURENCE HAMPTON, University of Oklahoma
W. L. HART, University of Minnesota.
W. W. HART, University of Wisconsin.
E. R. HEDRICK, University of California at Los Angeles.
T. H. HILDEBRANDT, University of Michigan.
T. R. HOLLCROFT, Wells College.
I. O. HORSFALL, L. D. S. College.
H. M. HOSFORD, Southern Methodist University
R. C. HUFFER, Beloit College.
JEWELL C. HUGHES, University of Arkansas.

LOUIS INGOLD, University of Missouri.
M. H. INGRAHAM, University of Wisconsin.

DUNHAM JACKSON, University of Minnesota.
C. M. JENSEN, Macalester College

DORA E. KEARNEY, Cedar Falls State Teachers College.
A. J. KEMPNER, University of Colorado.
E. C. KIEFER, James Millikin University.
W. C. KRATHWOHL, Armour Institute.

E. P. LANE, University of Chicago.
R. E. LANGER, University of Wisconsin.
C. G. LATIMER, University of Kentucky.

KURT LAVES, University of Chicago.
MAYME I. LOGSDON, University of Chicago.
W. D. MACMILLAN, University of Chicago.
J. V. MCKELVEY, Iowa State College.
A. S. MERRILL, University of Montana.
A. D. MICHAL, Ohio State University.
E. L. MICKLESON, University of Wisconsin.
C. N. MILLS, Illinois Normal University.
W. L. MISER, Vanderbilt University.
E. J. MOULTON, Northwestern University.

H. L. OLSON, Michigan State College.
F. W. OWENS, Pennsylvania State College.
HELEN B. OWENS, State College, Pa.

C. I. PALMER, Armour Institute.
G. A. PARKINSON, University of Wisconsin.
H. P. PETTIT, Marquette University.
T. A. PIERCE, University of Nebraska.

R. G. D. RICHARDSON, Brown University.
H. L. RIETZ, University of Iowa.
W. C. RISSELMAN, University of Minnesota.
E. D. ROE, Jr., Syracuse University.
W. H. ROEVER, Washington University.
W. E. ROTH, University of Wisconsin.
LULU RUNGE, University of Nebraska.

R. S. SHAW, Graduate School, University of Chicago.
W. A. SHEWHART, Bell Telephone Labs., New York, N. Y.
J. A. SHOHAT, University of Michigan.
MARY EMILY SINCLAIR, Oberlin College.
E. B. SKINNER, University of Wisconsin.
H. E. SLAUGHT, University of Chicago.
C. S. SLICHTER, University of Wisconsin.
G. W. SNEDECOR, Iowa State College.
VIRGIL SNYDER, Cornell University.
E. B. STOUFFER, University of Kansas.
K. D. SWARTZEL, University of Pittsburgh.

MARION TORREY, Goucher College.

E. B. VAN VLECK, University of Wisconsin.

I. N. WARNER, Platteville State Teachers College.
J. H. WEAVER, Ohio State University.
WARREN WEAVER, University of Wisconsin.
W. D. A. WESTFALL, University of Missouri.
ANNA PELL WHEELER, Bryn Mawr College.
W. R. WOODMANSEE, Ripon College.
EUPHEMIA R. WORTHINGTON, University of California at Los Angeles.

B. F. YANNEY, College of Wooster.
C. H. YEATON, Oberlin College.

The meetings were unusually pleasant and successful by reason of the conveniences afforded in the living arrangements and the manifold ways of entertaining those in attendance. The mathematicians and their friends for the most part had their living quarters and the meals in Chadbourne Hall, a central location with adequate social facilities and ready access to the lecture halls. On Wednesday afternoon two launches took a large party around Lake Mendota and to an afternoon tea at the summer home of Professor and Mrs. Slichter. Automobiles furnished by the Madison group took the visitors on Thursday afternoon about the many beautiful parts of Madison and vicinity and on Friday to the unique region of Devil's Lake. Smaller groups took fre-

quent advantage of the opportunities for golf, tennis, bathing, and canoeing. The many courtesies were recognized by a joint resolution offered at the banquet by Professor Hollcroft and adopted unanimously by a rising vote; this expressed the sincere thanks of the members to the administration, Professors Slichter and Van Vleck and the mathematical staff, the ladies, Professor Babcock and the committee on arrangements, the University Club and golf clubs, and the Press Bureau of the University.

The American Mathematical Society held its eleventh colloquium on Tuesday, Wednesday, Thursday and Saturday mornings and on Thursday evening, with lectures by Professor Anna Pell Wheeler on "The theory of quadratic forms in infinitely many variables and applications," and by Professor E. T. Bell on "Algebraic arithmetic." A large enrollment for these lectures assured the continued success of the Society's colloquium. The Society held a business session on Friday morning and sessions for the reading of papers on Thursday and Friday mornings.

160 persons attended the joint banquet held Wednesday evening at the Maple Bluff Golf Club under the eloquent toastmastership of Professor Slichter. A telegram of affectionate greeting was sent to Professor E. H. Moore by the secretaries on vote of those present; a reply was received the next morning expressing Professor Moore's grateful appreciation of the greetings and deep regret at his inability to attend the meetings. The banquet was unique in being addressed by three presidents. President Snyder of the Society told of the plans for the forthcoming meetings at Nashville, Amherst, and Bologna. President Ford of the Association spoke of the special field of mathematics characterized as the domain in which we lay down definitions of things; he emphasized the value of our passing over to society the means of clarifying social relations through clear adequate definitions. President Glenn Frank of the University of Wisconsin welcomed the mathematicians, expressing a high regard for their position in developing accuracy, clarity, and the creative imagination. The development of the sciences and the treatment of social problems are at heart, he said, one problem,—how can we make men outside and inside the laboratory? Can we find means to instill in our students the scientific spirit? It cannot be done simply as a by-product of our teaching. Shorter speeches were made by Professor Birkhoff on the new visiting mathematical professorships of the Society, by Professor Bell on the social contacts made in connection with these meetings, and by Professor Graves on the hope that the future holds in the field of mathematics.

The Mathematical Association held sessions on Monday and Tuesay afternoons with President Ford in the chair, Vice-President Kempner presiding for a part of the Tuesday session. The program was arranged by a committee consisting of Professors E. J. Moulton (chairman), Arnold Dresden, W. L. Hart,

W. B. Ford, and H. E. Buchanan. Abstracts of the papers are given, numbered in accordance with the numbers of the papers.

FIRST SESSION OF THE ASSOCIATION

(1) "Conditions for mathematical study in Italy" by Professor MAYME I. LOGSDON, University of Chicago.

(2) "A geometrical method for solving the biquadratic quation" by Professor W. C. GRAUSTEIN, Harvard University.

(3) "Machines for handling statistics" by Mr. F. A. HARPER, Tabulating Machine Company, Milwaukee, Wis., assisted by Mr. C. W. VAUGHN, chief disbursing officer of the University of Wisconsin.

1. The first part of the paper discussed informally reasons for going to Italy for mathematics, the curriculum and organization of the mathematical departments of the Italian universities, what the professors at Rome were doing in 1925–26, the libraries and Rome a Mecca for mathematicians. The second part outlined a portion of a theory of maxima and minima as developed by Steiner, Caratheodory, Schwarz, Zenadoro, Minkowski, and others, using instead of integration Cavalieri's method of indivisibles, a method "intermediate between the method of exhaustion of the Greeks and the later methods of the calculus."

Professors Stouffer and Lane, who have also visited and studied recently in Italy, agreed in saying that it is wise for Americans to have had their doctor's degree before going to Italy and to know quite clearly at what they wish to work, as the Italians are not so thoroughly acquainted with what has been accomplished in American mathematics and are accustomed to completing for themselves promising lines of research instead of giving these out to younger investigators as is done in America and other countries. The Italians nevertheless are most cordial to the Americans.

2. This paper will appear in an early issue of the Monthly.

Professor Snyder called special attention to the value of the last modification made in Professor Graustein's treatment, wherein the templet for $y^2-x=0$ can be turned through 90° and used as $x'^2-y'=0$; this he called a real contribution to the solution of the biquadratic.

3. After referring to the development of tabulating machines as described by Mr. Victor Johns in the issue of the Monthly for December, 1926, Mr. Harper said that mechanical accounting machines have unlimited uses: Insurance companies in general use them in the compilation of mortality and associated statistics, the United States and foreign governments apply them for compiling population, agricultural, manufactures, and vital statistics and for preparation of numerous exhibits shown in the survey of business and in sectional record books of business statistics prepared by the Department of Commerce. In

1900 the record of the capital, wages, and value of products of manufactures rose to figures almost beyond comprehension. The capital invested was $9,846,628,564; the salaries and wages paid amounted to $2,735,430,848; and the value of the products was $13,039,279,566. In agriculture the figures are almost equally impressive. The total value of farms in 1900 was $16,674,690,247 and that of agricultural products in 1899 was $4,739,118,752. To gather and collate such stupendous figures, not only with accuracy, but so swiftly that the record of population in 1900 appeared as quickly as did the little report of the first census, was a task of the first magnitude. Such a report could be prepared only with the help of modern tabulating machines.

Mechanical accounting equipment may be found, in a number of our largest colleges, for every-day use and for instructional purposes. The Universities of Wisconsin and Michigan, the La Salle Extension, Johns Hopkins, Cornell, and Columbia Universities are some of the users. The machines are used to prepare payrolls, to analyze direct labor charges, accounts payable and receivable, for research analysis work and for demonstration purposes in accounting and commercial courses.

Mr. Vaughn showed the actual use of the sorting and tabulating machines by means of a set of cards representing the classification and the totalling of individual expenditures of university funds among several special funds according to which classifications are desired.

SECOND SESSION OF THE ASSOCIATION

(4) "Mathematics for students of chemistry" by Professor FARRINGTON DANIELS, University of Wisconsin.

(5) "Mathematics for commerce students" by Professor E. B. SKINNER, University of Wisconsin. Discussion by Professor W. L. HART, University of Minnesota.

(6) "The Rhind mathematical papyrus" by Professor R. C. ARCHIBALD, Brown University.

4. This paper will appear in an early issue of the Monthly.

5. After a brief account of the way in which the course in investment theory came into existence at the University of Wisconsin, the author outlined what he considered a reasonable first course in the subject giving at the same tine what he deemed to be the necessary prerequisites. The foundations of the subject are the theory of interest and the theory of the annuity certain. Following these subjects the important problems that present themselves in the business world are three in number: (1) Amortization problems, (2) Bond value problems, (3) Depreciation and sinking fund problems.

The problems relating to building and loan association work are of less importance, mainly for the reason that they are less amenable to mathematical treatment.

Consideration was given to later work for the commerce student and to a number of problems which confront the teacher of the subject. The paper closed with a plea for the recognition of courses in mathematics for commerce students even though they promise little in the way of leading students toward advanced courses in mathematics.

Professor Hart said that Professor Skinner developed for the first time in the United States a course in the mathematics of investment, and mentioned his success in the selection of material for the detailed work of the course. The mathematics of this subject is essentially simple, the theory of annuities being the basis of the course; this simplicity should be emphasized by the teacher, along with the important applications of the theory.

6. Professor Archibald told of the discovery of the Rhind mathematical papyrus now in the British Museum and of certain of its fragments discovered in 1922 in the library of the New York Historical Society. He showed photographs of portions of the papyrus including a photograph, which had just been made of 36 of the 40 fragments put in place with reference to the ends of the two rolls of the papyrus in the British Museum. He referred also to seven mathematical documents besides the Rhind papyrus, which all belonged to the golden period of Egyptian science, 1900–1650 B.C. The processes in Egyptian mathematics and the contents of the papyrus were described in detail, with examples. Indications were given of the contents of Chancellor Chace's work on the Rhind papyrus to be published by the Association in December, 1927. Certain facts were mentioned with regard to the mathematical leather roll, B.M. 10250, which with the aid of a chemist was unrolled in the summer of 1927. The interesting announcement was made that this is to be reproduced in facsimile in the Chancellor's work.

MEETING OF THE BOARD OF TRUSTEES

Eleven trustees were present at the two sessions on Monday and Tuesday.

The following thirty-seven persons and one institution were elected to membership, on application duly certified.

To Individual Membership.

W. N. BARNES, B.S. (Chicago). Teacher, San Antonio, Texas.

A. K. BETTINGER, A.M. (Wisconsin). Instr., Creighton Univ., Omaha, Nebr.

SAM BYRD, M.S. (Arkansas). Teacher, Central High, Tulsa, Okla.

A. S. CROOM. Asst. Prof., Washburn College, Topeka, Kans.

ELINOR B. FLAGG, M.S. (Illinois). Asst. Prof., Illinois State Normal Univ., Normal, Ill.

W. H. GAGE, M.A. (Brit. Columbia). Instr., Victoria Coll., Victoria, B. C.

B. P. GILL, A.M. (Columbia). Instr., Coll. of City of New York, N. Y.

CHARLES HATFIELD, A.M. (Tennessee). Head of Dept., Lincoln Memorial Univ., Harrogate, Tenn.

EVELYN HESSELTINE, A.M. (Nebraska). Teacher, Spearfish Normal School, Spearfish, S. Dak.

H. K. HOLT, A.M. (Ohio State). Asst. Prof., Pennsylvania Military Coll., Chester, Pa.

C. M. JENSEN, Ph.D. (Minnesota). Asst. Prof., Macalester Coll., St. Paul, Minn.

LETTERIO LABOCCETTA, C.E. (Naples). Engineer, Rome, Italy.

R. E. LANGER, Ph.D. (Harvard). Prof., Univ. of Wisconsin, Madison, Wis.

A. J. LEWIS, A.M. (Denver). Instr., University of Denver, Denver, Colo.

E. L. MACKIE, Ph.D. (Chicago). Asso. Prof., Univ. of North Carolina, Chapel Hill, N. C.

NELLIE P. MYSER (Mrs. W. L.), A.B. (Huron). Teacher, College Math., Ward-Belmont, Nashville, Tenn.

MARY L. NEWTON, A.M. (Columbia). Dean, All Saints' College, Vicksburg, Miss.

BESS PATTON, M.S. (Chicago). Teacher, Girls High School, Atlanta, Ga.

R. E. PETERSON, B.S. (Grove City). Instr., Culver Military Acad., Culver, Ind.

E. J. PURCELL. Instr. in Math. and Physics, Evans School, Tucson, Ariz.

MARGARET RAMSEY, B.S. (Linfield). Instr., Linfield Coll., McMinnville, Ore.

D. M. RASEL, B.S. (Washington and Jefferson). Instr., Applied Math., Washington and Jefferson College, Washington, Pa.

W. J. REID, A.M. (Texas). Austin, Texas.

W. A. RICHARDS, A.M. (Chicago). Morton Jr. Coll., Cicero, Ill.

W. C. RISSELMAN, A.M. (Minnesota). Teaching Asst., Univ. of Minn., Minneapolis, Minn.

R. S. SHAW, B.S. (Bates). 5461 Univ. Ave., Chicago, Ill.

R. D. SMINK, M.S. (Bucknell). Teacher, High School, Lock Haven, Pa.

E. H. SNYDER, B.S. (Iowa). Industrial Engineer, Wis. Pearl Button Co., La Crosse, Wis.

J. R. K. STAUFFER, A.B. (Franklin and Marshal). Instr., Purdue Univ., LaFayette, Ind.

W. R. STEVENS. Meteorologist, U. S. Weather Bureau, Washington, D. C.

GUY STEVENSON, Ph.D. (Illinois). Asst. Prof., Univ. of Louisville, Louisville, Ky.

JOSEPHINE STONE, A.M. (Peabody). Teacher, Athens Coll., Athens, Ala.

F. E. ULRICH, B.S. in E.E. (Union). Instr., Union Coll., Schenectady, N. Y.

B. B. VANCE, B.S. (Univ. of Louisville). Instr., Univ. of Louisville, Louisville, Ky.

MILDRED WATT, A.M. (Cornell). Teacher, Buffalo Seminary, Buffalo, N. Y.

S. W. WILLIAMS, A.B., B.E. (Vanderbilt). Prof., Arkansas Coll., Batesville, Ark.

F. L. WREN, A.M. (Peabody). Asso. Prof., Teaching of Math., George Peabody Coll. for Teachers, Nashville, Tenn.

To Institutional Membership.

RHODE ISLAND COLLEGE OF EDUCATION, Providence, R. I.

Professor T. M. Focke and the Secretary-Treasurer were appointed as representatives of the Association on the Council of the American Association for the Advancement of Science.

The Trustees voted that the President and two others to be appointed by him be nominated as representatives of the Association on the American Section of the International Mathematical Union, in connection with the Congress to be held at Bologna, September 1–10, 1928. President Ford subsequently named Professors MAYME I. LOGSDON and R. C. ARCHIBALD.

It was voted to have the finance committee transfer an additional thousand dollars to the endowment fund and to purchase a bond for that amount. This committee was empowered to make any changes in the investments of the Association funds which they may think wise, it being understood that the investments should at all times be so made as to afford adequate protection for the Association.

A report on mathematical symbols was approved as made by the subcommittee on mathematical symbols of the American Engineering Standards

Committee, dated April 25, 1927. This report incorporates the present day procedure in elementary and higher mathematics and has been printed in several of the journals of the sponsoring engineering societies.

Brief discussion took place on the future meetings of the Association, progress made in the Carus Monograph plan and the mutual activities and co-operation of the sections and the parent body.

W. D. CAIRNS, *Secretary-Treasurer.*

THE CLASSIFICATION OF QUADRICS IN EUCLIDEAN N-SPACE BY MEANS OF COVARIANTS[1]

By PHILIP FRANKLIN, Massachusetts Institute of Technology

1. Introduction. The first *complete* classification in terms of invariants and covariants for the Cartesian equations of conics was recently given by Professor MacDuffee.[2] While his results leave nothing to be added, the method used does not seem to be that most natural for the problem. The treatment is from the standpoint of Lie, whereas the problem is purely algebraic. Also the method gives no suggestion as to what the covariants would be for higher dimensions,[3] and would be almost inapplicable if these covariants were not already known from other considerations. Consequently it seems desirable to give a purely algebraic treatment of the question, which generalizes at once to n-space. This is done in the present paper.

2. Invariants of a conic. For the sake of completeness, as well as to pave the way for generalization, we give here a derivation of the well-known invariants of a second degree polynomial

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = \phi(x, y) \tag{1}$$

under the Euclidean transformations

$$x = b_{11}x' + b_{12}y' + b_{13}, \quad y = b_{21}x' + b_{22}y' + b_{23}, \tag{2}$$

where

$$\begin{Vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}. \tag{3}$$

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[1] Presented to the American Mathematical Society, October 30, 1926.

[2] C. C. MacDuffee, *Euclidean invariants of second degree curves*, this Monthly, vol. 33 (1926), pp. 243–252.

[3] Thus the application to 3-space requires a new investigation, equal in length to the old. Cf. L. J. Paradiso, *A classification of second degree loci of space*, this Monthly, vol. 33 (1926), pp. 406–418.

The matrix form of (1) is

$$(4)\qquad \| x \quad y \quad 1 \| \cdot \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ 1 \end{Vmatrix} = \phi(x,y)\ ; \quad (a_{ij} = a_{ji}).$$

We may write (2) in either of the forms

$$(5)\qquad \| x \quad y \quad 1 \| = \| x' \quad y' \quad 1 \| \cdot \begin{Vmatrix} b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & 0 \\ b_{13} & b_{23} & 1 \end{Vmatrix}$$

or

$$(6)\qquad \begin{Vmatrix} x \\ y \\ 1 \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x' \\ y' \\ 1 \end{Vmatrix}.$$

Using these last two equations, we find as the transform of (4)

$$(7)\qquad \| x' \quad y' \quad 1 \| \cdot \begin{Vmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{Vmatrix} \cdot \begin{Vmatrix} x' \\ y' \\ 1 \end{Vmatrix} = \phi'(x',y')\ ,$$

where

$$(8)\qquad \begin{Vmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} & 0 \\ b_{12} & b_{22} & 0 \\ b_{13} & b_{23} & 1 \end{Vmatrix} \cdot \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{Vmatrix}.$$

As this matrix equation implies the corresponding determinant equation, and as the product of the two determinants in the b's is unity from our orthogonality requirement (3), we have

$$(9)\qquad D_3 = |a_{ij}| = |a'_{ij}|, \text{ as the first invariants.}$$

Also, from the way the zeros occur in (8), that equation implies

$$(10)\qquad \begin{Vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{Vmatrix} \cdot \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

But, from (3), we have for all values of k,

$$(11)\qquad \begin{Vmatrix} k & 0 \\ 0 & k \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{Vmatrix} \cdot \begin{Vmatrix} k & 0 \\ 0 & k \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

On subtracting the terms of (11) from those of (10), we obtain :

$$(12)\qquad \begin{Vmatrix} a'_{11}-k & a'_{12} \\ a'_{21} & a'_{22}-k \end{Vmatrix} = \begin{Vmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{Vmatrix} \cdot \begin{Vmatrix} a_{11}-k & a_{12} \\ a_{21} & a_{22}-k \end{Vmatrix} \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}.$$

As this matrix equation implies the corresponding determinant equation, and since the product of the two determinants in the b's is unity by (3), we have

$$(13)\qquad \begin{Vmatrix} a_{11}-k & a_{12} \\ a_{21} & a_{22}-k \end{Vmatrix} = k^2 - D_1 k + D_2$$

is invariant. As this is so for all values of k, the coefficients must separately be invariant.

We thus have as the *invariants* of the polynomial (1), D_3, D_2 and D_1, whose *weights* or degrees in the coefficients are given by their subscripts. Their ratios are invariants for the locus of points for which $\phi(x, y)=0$, and have a geometric significance which we shall give later.

3. Covariants of a conic. To obtain the covariants of the second degree polynomial (1), we shall find it convenient to introduce line coördinates. We first transform to homogeneous point coördinates, z_1, z_2, z_3, where $x=z_1/z_3$, $y=z_2/z_3$, so that the equation of our conic, $\phi(x, y)=0$ becomes :

$$(14)\qquad \sum_i \sum_j a_{ij} z_i z_j = 0.$$

Let us temporarily assume $D_3 = |a_{ij}| \neq 0$; we shall remove this restriction later. We find as the equation of the tangent to (14) at z_1, z_2, z_3,

$$(15)\qquad \sum_i \sum_j a_{ij} Z_i z_j = 0,$$

where Z_1, Z_2, Z_3 denote current coördinates. The coördinates of this line in homogeneous line coördinates are

$$(16)\qquad u_i = \sum_j a_{ij} z_j,$$

since the equation in point coördinates Z_i, of a line with line coördinates u_i is

$$(17)\qquad \sum_i u_i Z_i = 0.$$

Solving equations (16) for z_j, and using A_{ij} to denote the co-factor of a_{ij} in $A = D_3 = |a_{ij}|$, we find :

$$(18)\qquad z_j = \sum_i u_i A_{ij}/A,$$

and on inserting this in (14) we get

$$(19) \qquad \sum_i \sum_p \sum_q \sum_j \frac{a_{ij}u_p A_{pi}u_q A_{qj}}{A^2} = \sum_q \sum_j \frac{A_{qj}u_q u_j}{A} = 0.$$

The relation between (14) and (19) is reciprocal, as was to be expected from the duality relation of point to line coördinates. We note in passing that if (19) is used, multiplying the left member of the point equation by a constant is equivalent to dividing the left member of the line equation by this constant. If we start with the line equation :

$$(20) \qquad \sum_i \sum_j s_{ij}u_i u_j = \sum_i \sum_j A_{ij}u_i u_j/A = 0,$$

we may write the point equation as[1,2]

$$(21) \qquad \sum_i \sum_j a_{ij}z_i z_j = \sum_i \sum_j S_{ij}z_i z_j/S = 1/S \begin{vmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & s_{11} & s_{12} & s_{13} \\ z_2 & s_{21} & s_{22} & s_{23} \\ z_3 & s_{31} & s_{32} & s_{33} \end{vmatrix}.$$

Here $S = |s_{ij}|$ and S_{ij} is the cofactor of s_{ij} in S.

We now wish to consider the equations of transformation for the z_i and u_i. In view of (2) and (6), we have for the z_i :

$$(22) \qquad \begin{Vmatrix} z_1 \\ z_2 \\ z_3 \end{Vmatrix} = n \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} z_1' \\ z_2' \\ z_3' \end{Vmatrix}.$$

As the u_i are defined by (17), we must have

$$(23) \qquad \sum_i u_i' z_i' = m \sum_i u_i z_i,$$

where m is a constant multiplier or in matrix form

$$(24) \qquad \|u_1' \quad u_2' \quad u_3'\| \cdot \begin{Vmatrix} z_1' \\ z_2' \\ z_3' \end{Vmatrix} = m\|u_1 \quad u_2 \quad u_3\| \cdot \begin{Vmatrix} z_1 \\ z_2 \\ z_3 \end{Vmatrix}.$$

By combining this with (22), we see that the law of transformation of the u_i is

$$(25) \qquad \|u_1' \quad u_2' \quad u_3'\| = mn\|u_1 \quad u_2 \quad u_3\| \cdot \begin{Vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{Vmatrix}.$$

[1] For the properties of minors here used, cf. Bôcher, *Higher Algebra*, p. 30.

[2] For the dual coördinates, cf. G. Salmon, *Analytic Geometry of Three Dimensions*, revised by A. P. Rogers, 5th ed., 1912, vol. 1, p. 65, p. 126.

We note that, in consequence of this,

$$\|u_1' \quad u_2'\| = mn\|u_1 \quad u_2\| \cdot \begin{Vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{Vmatrix}. \tag{26}$$

From this, in view of (3), we derive

$$u_1'^2 + u_2'^2 = m^2n^2(u_1^2 + u_2^2). \tag{27}$$

Now consider the equation

$$\sum_i \sum_j s_{ij}u_iu_j - k(u_1^2 + u_2^2) = 0, \tag{28}$$

where k is any number. On transforming the coördinates by (25), this equation becomes

$$(1/m^2n^2)\left[\sum_i \sum_j s_{ij}'u_i'u_j' - k(u_1'^2 + u_2'^2)\right] = 0. \tag{29}$$

By (21), the point form of (28) is :

$$\frac{1}{S}\begin{vmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & s_{11} - k & s_{12} & s_{13} \\ z_2 & s_{21} & s_{22} - k & s_{23} \\ z_3 & s_{31} & s_{32} & s_{33} \end{vmatrix} = A \begin{vmatrix} 0 & z_1 & z_2 & z_3 \\ z_1 & \frac{A_{11}}{A} - k & \frac{A_{12}}{A} & \frac{A_{13}}{A} \\ z_2 & \frac{A_{21}}{A} & \frac{A_{22}}{A} - k & \frac{A_{23}}{A} \\ z_3 & \frac{A_{31}}{A} & \frac{A_{32}}{A} & \frac{A_{33}}{A} \end{vmatrix} = 0. \tag{30}$$

The point form of (29), after removal of the factor m^2n^2, will only differ from this in having primes on the A's. As these two point equations represent the same locus, their left members will differ at most by a factor. If we form the corresponding Cartesian equations by replacing z_1, z_2, z_3 by x, y, 1, respectively, the highest power of k in each expression will be the same, since $A = A' = D_3$, so that the factor is unity, and

$$-A\begin{vmatrix} 0 & x & y & 1 \\ x & \frac{A_{11}}{A} - k & \frac{A_{12}}{A} & \frac{A_{13}}{A} \\ y & \frac{A_{21}}{A} & \frac{A_{22}}{A} - k & \frac{A_{23}}{A} \\ 1 & \frac{A_{31}}{A} & \frac{A_{32}}{A} & \frac{A_{33}}{A} \end{vmatrix} = Ak^2 - C_2k + C_1 \tag{31}$$

will be equal to the corresponding expression in the primed variables. As k is arbitrary, the coefficients may be separately equated, and hence are covariants.

Using known properties of determinants, (see p. 456, ftn. 1) we may write them explicitly as

$$C_2 = (x^2 + y^2)A_{33} - 2xA_{13} - 2yA_{23} + A_{11} + A_{22}, \qquad C_1 = \phi(x, y). \tag{32}$$

Thus the apparent denominator cancels, and they are polynomials in the x, y and coefficients, of weight in the latter equal to their subscripts. The degree of C_2 in x, y may fall for particular polynomials, but is independent of coördinates owing to the linear character of (2).

While the proof of the covariance of C_1 and C_2 just given only applies if $A \neq 0$, since these expressions are polynomials in the coefficients, continuity considerations show that they must also be covariants when $A = 0$.

The equation $C_1 = 0$ represents the original locus; the geometrical significance of $C_2 = 0$ will be given later.

4. Classification of conics. We are now in a position to give a complete classification of the loci given by $\phi(x, y) = 0$. We begin by recalling the theorem that a real quadratic form in n variables of rank r can be reduced by a real orthogonal transformation in the n variables to the form[1]

$$c_1x_1^2 + c_2x_2^2 + \cdots + c_rx_r^2. \tag{33}$$

This shows that, by taking $b_{13} = b_{23} = 0$, and suitable values of the remaining coefficients in (2), i.e. by a rotation, we may reduce (1) to the form:

$$a_{11}x^2 + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33}. \tag{34}$$

If $a_{11} \neq 0$, and $a_{22} \neq 0$, a translation of axes gives the form

$$a_{11}x^2 + a_{22}y^2 + a_{33}. \tag{35}$$

If in (34) a_{11} and a_{22} were both zero, we would not have a conic, but a first degree equation, so that this case can not occur. If either one is zero, we take it as a_{22} and, by a translation, reduce (34) to the form

$$a_{11}x^2 + 2a_{23}y + a_{33}. \tag{36}$$

Unless $a_{23} = 0$, we may by a further translation make $a_{33} = 0$.

Thus we are led to the following cases:

I. $D_2 \neq 0$. Hence $a_{11} \neq 0$, $a_{22} \neq 0$, and $\phi(x, y)$ may be reduced to

$$C_1 = a_{11}x^2 + a_{22}y^2 + a_{33}. \tag{37}$$

[1] Cf. Bôcher, *Higher Algebra*, pp. 171–173.

For this case (13) becomes

$$(a_{11} - k)(a_{22} - k) = k^2 - D_1 k + D_2, \tag{38}$$

so that a_{11} and a_{12} are determined from our invariants as the zeros of this expression in k, while from (9), $a_{33} = D_3/D_2$. We may also solve for x^2 and y^2 in terms of our invariants and covariants, unless $a_{11} = a_{22}$. For, in this case, (31) becomes

$$-a_{11}a_{22}a_{33}\begin{vmatrix} 0 & x & y & 1 \\ x & \dfrac{1}{a_{11}} - k & 0 & 0 \\ y & 0 & \dfrac{1}{a_{22}} - k & 0 \\ 1 & 0 & 0 & \dfrac{1}{a_{33}} \end{vmatrix} = Ak^2 - C_2 k + C_1. \tag{39}$$

On putting $k = 1/a_{11}$, this gives

$$x^2(a_{11} - a_{22}) = A(1/a_{11})^2 - C_2(1/a_{11}) + C_1, \tag{40}$$

which determines x^2, and on putting $k = 1/a_{22}$ it gives

$$y^2(a_{22} - a_{11}) = A(1/a_{22})^2 - C_2(1/a_{22}) + C_1, \tag{41}$$

which determines y^2, in terms of the a_{ii}, which have already been shown to be given by the invariants, and the covariants.

If $a_{11} = a_{22}$, we can only solve for $x^2 + y^2$. We do this by differentiating (39) with respect to k, and then putting $k = 1/a_{11} = 1/a_{22}$, obtaining:

$$-a_{11}a_{22}(x^2 + y^2) = 2A(1/a_{11}) - C_2. \tag{42}$$

II. $D_2 = 0$, $D_3 \neq 0$. Hence $a_{22} = 0$, $a_{23} \neq 0$, and $\phi(x, y)$ may be reduced to

$$C_1 = a_{11}x^2 + 2a_{23}y. \tag{43}$$

For this case (13) becomes

$$-k(a_{11} - k) = k^2 - D_1 k, \tag{43}$$

which determines a_{11}, while $a_{23}^2 = -D_3/D_1$. We may solve for x^2 in terms of our invariants and covariants by evaluating (31) for this case, and putting $k = 1/a_{11}$. There results

$$a_{11}x^2 = A(1/a_{11})^2 - C_2(1/a_{11}) + C_1. \tag{44}$$

We obtain y by combining this with (43).

III. $D_2 = 0$, $D_3 = 0$. Hence $a_{22} = a_{23} = 0$, and $\phi(x, y)$ may be reduced to

$$C_1 = a_{11}x^2 + a_{33}. \tag{45}$$

Here (13) again takes the form (43), from which $a_{11}=D_1$, while (32) shows that C_2 is here independent of x and y, and $a_{33}=C_2/D_1$. We may solve for x^2 in terms of our invariants and covariants by using (45).

From the discussion just given, we may draw the following conclusions. If the invariants D_1, D_2 and D_3 are known, we may at once determine which of the three cases is applicable, and hence the appropriate canonical form. The coefficients of this canonical form are given, in cases I and II in terms of these invariants; in case III, in terms of these invariants and C_2. As any invariant of the given polynomial may be calculated from the coefficients of the canonical form, it must be a function of such invariants as determine these coefficients. Thus, in cases I and II, D_1, D_2, D_3 form a *complete system* of algebraic invariants. In case III, D_1, D_2, D_3 and C_2 form a complete system.

We further note that any covariant of the given polynomial would, for the canonical form, reduce to a function of the coefficients and variables in this canonical form, and such a function of the variables as to be unchanged by any transformation of coördinates which left the canonical form unchanged. Thus in case I, the covariant could only involve x^2 and y^2, as (37) is unchanged by $x=-x'$, or $y=-y'$. If $a_{11}=a_{22}$, (37) would be unchanged by the rotations which leave x^2+y^2 unchanged, so that the covariant would only involve x and y in this combination. Similarly in case II, a covariant could only involve x^2, and y. In case III, the covariant could only involve x^2, and must be independent of y, since $y=y'+b_{23}$ leaves (45) unchanged. Since each of these combinations of the variables has been calculated in terms of the invariants and covariants for the case in question, it follows that D_1, D_2, D_3, C_1, C_2 form a *complete system* of algebraic invariants and covariants for the given ploynomial. That the system is complete in the sense of having the minimum number follows from the number of coefficients and variables in the canonical terms, which are all independent.

It might be supposed that the discussion here used to verify that we have the right number of invariants and covariants was unnecessary since an application of the Lie theory, apparently merely requires us to count the number of variables in our form and parameters in the group and subtract. However, to justify this process we would have actually to set up the generators of the group explicitly in order to verify that the differential equations obtained by equating them to zero are independent.[1]

[1] The system of differential equations obtained by equating to zero the generators of an r-parameter group if independent, form a complete system, but are not always independent. (cf. E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 4.). The fact that equations (6), MacDuffee, l.c. p. 246 and (2) Paradiso, l.c. p. 407 are independent does not follow from the Lie theory, a priori, but must be proved, either by setting up and inspecting the equations, as these authors have done, or by arguing as we have done here.

5. Geometric considerations. To connect a second degree polynomial in x and y with a geometric locus, we equate the polynomial to zero. As the locus is unchanged when the polynomial is multiplied by a non-vanishing constant, only invariants and covariants of weight zero in the coefficients are geometric invariants or covariants. From the five invariants and covariants already found, by taking ratios, we may form four invariants and covariants of weight zero, which form a complete system of geometric invariants and covariants. The significance of the invariants varies with the case, but may be found from the canonical form.

Thus for central conics, case I, $D_3 \neq 0$, the reciprocals of the squares of the semi-axes are a_{11}/a_{33} and a_{22}/a_{33} and hence by (38) the ratios D_1D_2/D_3 and D_2^3/D_3^2 are the symmetric functions of these reciprocals. For degenerate central conics, case I, $D_3 = 0$, the ratios of the reciprocals of the squares of the semi-axes to the product of these reciprocal semi-axes for central conics having the degenerate one as a limit have significance, and the symmetric functions of these ratios are given by $D_1/D_2^{1/2}$ and 1.

For parabolas, case II, the reciprocal of the square of the semi-chord at unit distance from the vertex is $a_{11}/2a_{23}$, and hence by (43) is $\sqrt{(-D_1^3/4D_3)}$.

While only the ratios of the invariants have geometric significance the vanishing of any one of them has a direct geometric meaning. Thus from the classification it follows that $D_3 = 0$ means the locus degenerates into two straight lines. $D_2 = 0$ means that the locus at infinity degenerates giving a single point there either for the parabola of case II, or the parallel lines of case III. For $D_1 = 0$, we must have $D_2 \neq 0$ and from (37) we see that either the locus is degenerate, and consists of two perpendicular lines, or it is a central conic with perpendicular asymptotes.

Since the covariants represent the symmetric functions of the zeros of (31) we must interpret the meaning of these zeros to see the significance of the covariants. Each of these values of k, when inserted in (28) determines a conic passing through x, y, confocal to the given one, since it has the same tangents from the circular points at infinity.[1] These two conics are orthogonal. Let us introduce homogeneous line coördinates, and take the normals to these conics as axes. The first conic, by (28), has as its equation :

$$\sum_i \sum_j s_{ij}u_iu_j - k_1(u_1^2 + u_2^2) = 0. \tag{46}$$

With this choice of coördinates, the x axis, $(1, 0, 0)$, is a tangent, which necessitates $s_{11} = k_1$, and the equation of its point of contact

$$\sum_i s_{i1}u_1 - k_1u_1 = 0 \tag{47}$$

[1] Salmon-Rogers, l.c., p. 177.

must reduce to the equation of the origin, $u_3=0$. Thus $s_{12}=0$. A similar argument for the second conic shows that $s_{22}=k_2$ and $s_{23}=0$, so that the equation of the given conic is :

$$k_1u_1^2 + k_2u_2^2 + 2s_{13}u_1u_3 + 2s_{23}u_2u_3 + s_{33}u_3^2 = 0. \tag{48}$$

The tangent cone from the origin is the intersection of this with $u_3=0$, and has the equation :

$$k_1u_1^2 + k_2u_2^2 = 0. \tag{49}$$

The point form is

$$x^2/k_1 + y^2/k_2 = 0. \tag{50}$$

This shows that $C_2=0$ is the equation of the locus of points from which two perpendicular tangents can be drawn to the conic, while $C_1=0$, the given locus is that of points from which the two tangents coincide.

6. Second degree loci in n-space. Let us now consider the second degree polynomial in n variables $(i, j=1, 2, \cdots, n)$,

$$\sum_i \sum_j a_{ij}x_ix_j + 2\sum_i a_{i,n+1}x_i + a_{n+1,n+1} = \phi(x_1, x_2, \cdots, x_n). \tag{51}$$

We regard the x_i as coördinates in n-space, transformed by

$$x_i = \sum_j b_{ij}x_j' + b_{i,n+1}, \tag{52}$$

the corresponding homogeneous transformation being orthogonal. The equation $\phi(x_i)=0$ is thus the Cartesian equation of a quadric in Euclidean n-space.

The argument of section 2 at once generalizes and leads us to the invariants $D_1, D_2, \cdots, D_{n+1}$ of weight in the coefficients of (51) given by their subscripts, defined by :

$$D_{n+1} = |a_{ij}| \tag{53}$$

and

$$\begin{vmatrix} a_{11}-k & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-k & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn}-k \end{vmatrix} = (-k)^n + D_1(-k)^{n-1} + D_2(-k)^{n-2} + \cdots + D_n. \tag{54}$$

The arguments of section 3 may also be applied here, using as the dual coördinates $(n-1)$-space coördinates, and give the covariants $C_1, C_2, \cdots, C_n$, with weights equal to their subscripts, defined by :

$$(55)\qquad -A\begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n & 1 \\ x_1 & \dfrac{A_{11}}{A}-k & \dfrac{A_{12}}{A} & \cdots & \dfrac{A_{1n}}{A} & \dfrac{A_{1,n+1}}{A} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ x_n & \dfrac{A_{n1}}{A} & \dfrac{A_{n2}}{A} & \cdots & \dfrac{A_{nn}}{A}-k & \dfrac{A_{n,n+1}}{A} \\ 1 & \dfrac{A_{n+1,1}}{A} & \dfrac{A_{n+2,2}}{A} & \cdots & \dfrac{A_{n+1,n}}{A} & \dfrac{A_{n+1,n+1}}{A} \end{vmatrix}$$

$$= A(-k)^n + C_n(-k)^{n-1} + C_{n-1}(-k)^{n-2} + \cdots + C_1.$$

The apparent denominator cancels (see p. 456. ftn. 1), the C_i being polynomials in x_i and the a_{ij}, of the second degree in the x_i, except for special given polynomials, and of weight in the a_{ij} equal to their subscripts.

To classify the canonical forms to which $\phi(x_i)$ may be reduced, we first note that by the theorem quoted at the beginning of section 4, there exists a homogeneous orthogonal transformation (52) which reduces (51) to the form

$$(56)\qquad \sum_i a_{ii}x_i^2 + 2\sum_i a_{n+1,i}x_i + a_{n+1,n+1}.$$

If $a_{ii}\neq 0$, $i=1, 2, \cdots, n$, a translation of the axes reduces this to

$$(57)\qquad \sum_i a_{ii}x_i^2 + a_{n+1,n+1}.$$

If $n-r$ of the $a_{ii}=0$, we may take them as the last $n-r$, and by a translation of the axes reduce (56) to

$$(58)\qquad \sum_{s=1}^{r} a_{ss}x_s^2 + 2\sum_{t=r+1}^{n} a_{n+1,t}x_t + a_{n+1,n+1}.$$

If the $a_{n+1,t}$ are not all zero, we may take as the new x_{r+1} coördinate $n-1$ space that whose equation is

$$(59)\qquad 2\sum_{t=r+1}^{n} a_{n+1,t}x_t + a_{n+1,n+1} = 0, \text{ and (58) takes the forms}$$

$$(60)\qquad \sum_{s=1}^{r} a_{ss}x_s^2 + 2a_{r+1,n+1}x_{r+1}.$$

If all the $a_{n+1\ t}=0$, (58) reduces to

$$(61)\qquad \sum_{s+1}^{r} a_{ss}x_s^2 + a_{n+1,n+1}.$$

We may now consider the cases which can occur.

I. $D_n \neq 0$. Hence in (56) $a_{ii} \neq 0$, $i = 1, 2, \cdots, n$ and $\phi(x_i)$ may be reduced to

$$C_1 = \sum_i a_{ii}x_i^2 + a_{n+1,n+1}. \tag{62}$$

For this case (54) becomes

$$(a_{11} - k)(a_{22} - k) \cdots (a_{nn} - k) = (-k)^n + D_1(-k)^{n-1} + D_2(-k)^{n-2} + \cdots + D_n, \tag{63}$$

so that the a_{ii} are determined in terms of our invariants, while from (53) $a_{n+1\,n+1} = D_{n+1}/D_n$. To solve for x_i^2 in terms of our invariants and covariants, we use (62) to reduce (55), and put $k = A_{ii}/A = 1/a_{ii}$, which reduces (55) to the form

$$\frac{x_i^2}{a_{ii}^{n-2}}(a_{ii} - a_{11}) \cdots (a_{ii} - a_{i-1\,i-1})(a_{ii} - a_{i+1\,i+1}) \cdots (a_{ii} - a_{nn}) = A(-1/a_{ii})^n + C_n(-1/a_{ii})^{n-1} + C_{n-2}(-1/a_{ii})^{n-2} + \cdots + C_1. \tag{64}$$

If p of the a_{ii} are equal, the corresponding x_i^2 can no longer be determined by (64).

The sum of these p x_i^2 may be determined, however, by differentiating (55) $p-1$ times with respect to k before putting $k = 1/a_{11} = 1/a_{2\cdot} = \cdots 1/a_{pp}$, and thus obtaining a formula analogous to (42).

II. $D_n = 0$, $D_{n+1} \neq 0$. Hence in (56) $a_{nn} = 0$, $a_{n+1,n} \neq 0$, and, by (60), $\phi(x_i)$ may be reduced to

$$\sum_{s=1}^{n-1} a_{ss}x_s^2 + 2a_{n,n+1}x_n. \tag{65}$$

For this case (54) becomes

$$-k(a_{11} - k) \cdots (a_{n-1,n-1} - k) = (-k)^n + D_1(-k)^{n-1} + \cdots + D_{n-1}(-k), \tag{66}$$

which determines the a_{ss}, while $a_{n,n+1}^2 = -D_{n+1}/D_{n-1}$. To solve for x_s^2, we reduce (55) by using (65) and putting $k = 1/a_{ss}$. This leads to

$$\frac{x_s}{a_{ss}^{n-3}}(a_{ss} - a_{11}) \cdots (a_{ss} - a_{s-1,s-1})(a_{ss} - a_{s+1,s+1}) \cdots (a_{ss} - a_{n-1,n-1}) = A(-1/a_{ss})^n + C_n(-1/a_{ss})^{n-1} + C_{n-2}(-1/a_{ss})^{n-2} + \cdots + C_1. \tag{66}$$

If p of the a_{ss} are equal, the corresponding x_s^2 can no longer be determined by (66), but by differentiating (55) $p-1$ times with respect to k before substituting $1/a_{ss}$ for k, the sum of these p x_s^2 may be determined.

III. $D_{n+1} = D_n = 0$. Hence in (56), by the method used to obtain (60) and (61) we may make $a_{nn} = a_{n,n+1} = 0$.

Here the right member of (55) is a determinate polynomial in the a_{ij}, but can no longer be directly evaluated from the left member, owing to the vanishing denominator. One simple way of evaluating this left member in this case, where all the $a_{ni}=0$, is to first put $a_{ni}=0$, $i \neq n$, $a_{nn}\neq 0$, and find the limit when a_{nn} approaches zero. If we put $A=a_{nn}A'$, and $a_{ni}=0$, $i\neq n$, we shall have $A_{ij}=a_{nn}A'_{ij}$, $(i, j\neq n)$, and may write the left member of (55) in the form:

$$(67)\qquad -A'\begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n & 1 \\ x_1 & \dfrac{A'_{11}}{A'}-k & \dfrac{A'_{12}}{A'} & \cdots & 0 & \dfrac{A'_{1,n+1}}{A'} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ a_{nn}x_n & 0 & 0 & \cdots & 1-a_{nn}k & 0 \\ 1 & \dfrac{A'_{n+1,1}}{A'} & \dfrac{A'_{n+1,2}}{A'} & \cdots & 0 & \dfrac{A'_{n+1,n+1}}{A'} \end{vmatrix}.$$

The limit of this when a_{nn} approaches zero is

$$(68)\qquad -A'\begin{vmatrix} 0 & x_1 & x_2 & & x_{n-1} & 1 \\ x_1 & \dfrac{A'_{11}}{A'}-k & \dfrac{A'_{12}}{A'} & \cdots & \dfrac{A'_{1,n-1}}{A'} & \dfrac{A'_{1,n+1}}{A'} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ x_{n-1} & \dfrac{A'_{n-1,1}}{A'} & \dfrac{A'_{n-1,2}}{A'} & \cdots & \dfrac{A'_{n-1,n-1}}{A'}-k & \dfrac{A'_{n-1,n+1}}{A'} \\ 1 & \dfrac{A'_{n+1,1}}{A'} & \dfrac{A'_{n+1,2}}{A'} & \cdots & \dfrac{A'_{n+1,n}}{A'} & \dfrac{A'_{n+1,n+1}}{A'} \end{vmatrix}$$

$$=A'(-k)^{n-1}+C'_{n-1}(-k)^{n-2}+C'_{n-2}(-k)^{n-3}+\cdots+C'_1.$$

This is essentially the same form as (55) except for the primes and the numbering $1, 2, \cdots, n-1, n+1$ instead of $1, 2, \cdots, n, n+1$. A comparison with (55) shows that $C_j'=C_j$, $j=1, 2, \cdots, n-1$ while $C_n=A'$, so that this particular covariant is constant in this and hence in all coördinate systems and reduces to an invariant.

On putting $a_{ni}=0$ $(i=1, 2, \cdots, n)$ in (54), it becomes

$$(69)\qquad \begin{vmatrix} a_{11}-k & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22}-k & \cdots & a_{2,n-1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n-1,1} & a_{n-2,2} & \cdots & a_{n-1,n-1}-k \end{vmatrix}$$

$$=(-k)^{n-1}+D_1(-k)^{n-2}+D_2(-k)^{n-3}+\cdots+D_{n-1}.$$

This is of the same form as (54) in one less variable.

As we have found for this case coördinates such that one of them, x_n is missing from the given equation, the locus in this case is a "cylinder" in n space. The right section in the space $x_i (i=1, 2, \cdots, n-1)$ is a quadric in this space, which may be classified into two cases, analogous to I and II above. The coefficients and variables in the canonical forms are obtained by the same method previously used, except that equations (54) and (55) are replaced by (69) and (68) respectively.

We have the analogue of case I if $D_{n-1} \neq 0$, and that of case II if $C_n \neq 0$. If $D_{n-1}=0$ and $C_n \equiv 0$, certain of our formulas will contain the vanishing factor A' in the denominator, but in this case we may repeat the process used to obtain (68) and (69). Evidently if our given equation involves essentially $n-r$ variables we need to repeat the process r times. There will be $n-r$ true covariants, the remaining r will reduce to invariants for this case.

By the reasoning used at the end of section 4, we see that the invariants, together with certain of the covariants in special cases, form a complete system of algebraic invariants for the given polynomial. The set of invariants and covariants we have found form a complete system of algebraic invariants and covariants for this polynomial.

7. Geometric considerations. As in section 5, we see that if we form, from our $n+1$ invariants and n covariants, $2n$ ratios of weight zero, we shall have a complete system of geometric invariants and covariants for the locus of $\phi(x_i)=0$. The geometric significance of these ratios is readily obtained from the canonical form.

Thus in case I, we have central n-dimensional quadrics, and the symmetric functions of the reciprocals of the squares of their semi axes, $a_{ii}/a_{n+1,n+1}$, are $D_i D_n^i/(D_{n+1})^i$, by (63). For degenerate central quadrics, n-dimensional cones, the ratios of the reciprocals of the squares of the semi-axes to the nth root of the product of these reciprocal squares for central quadrics having the degenerate ones as a limit have significance, and the symmetrical functions of these ratios are given by $D_i/(D_n)^{i/n}$.

For n-dimensional paraboloids, the $a_{ii}/a_{n,n+1}$ are the reciprocals of the squares of the semiaxes of a section at unit distance from the vertex, and their symmetric functions are by (66) $D_i(-D_{n-1}/D_{n+1})^{i/2}$.

Analogous considerations apply to the subdivisions of case III.

The vanishing of the invariants has a geometric interpretation. Thus $D_{n+1}=0$, if $D_n \neq 0$, means that the locus is an n-dimensional cone, while $D_n=0$, means that the locus at infinity degenerates into a cone. If our locus is an n-dimensional cone, the condition $D_i=0$, $i=1, 2, \cdots, n-1$, means that there exist $n!/i!\,(n-i)!$ mutually perpendicular i-spaces, each tangent to this cone. For, suppose such a set of spaces exist, and take them as the coördinate i-spaces. Since the space $x_{i+1}=x_{i+2}=\cdots=x_n=0$ is tangent to the cone, on putting these

values in the equation, the remaining terms must represent this i-space counted twice, i.e., must factor. Hence, regarding these terms as the equation of a cone in i-space, this cone must cut the $i-1$ space at infinity in a degenerate locus, and for it $D_i' = 0$. That is, a principal i-rowed minor of D_n vanishes and by similar reasoning all such principal i-rowed minors vanish. But, by (54) D_i is the sum of these minors, and hence vanishes, for the particular coördinates used. As it is invariant, it vanishes in all systems. This proves the necessity of the condition $D_i = 0$ for the cone's having the maximum number of mutually perpendicular tangent i-spaces. From the algebraic formulation of the problem, it appears that there is only one equation giving the necessary and sufficient condition, so that the condition $D_i = 0$ is also sufficient. For a central quadric not a cone, the condition $D_i = 0$ means that the asymptotic cone has the above property.

The interpretation of the covariants hinges on the meaning of k in (55). Each value of k which makes the members of (55) vanish, when inserted in the equation

$$(70) \quad \sum_i \sum_j s_{ij} u_i u_j - k \sum_{s=1}^{n} u_s^2 = 0 ; \quad [s_{ij} = A_{ij}/A ; \quad i,j = 1,2, \cdots, n+1],$$

analogous to (28), gives, in homogeneous dual $n-1$ space coördinates, the equation of a quadric passing through the point x_i confocal to the given one. The normals to these n quadrics at the point x_i are mutually perpendicular, and taking them as coordinate axes, by an argument (see p. 461, ftn. 1) similar to that of section 5, we find for the dual equation of the tangent cone from the origin to our original quadric

$$\sum_{s=1}^{n} k_s u_s^2 = 0, \quad \text{and as its point form} \quad \sum_{s=1}^{n} x_s^2/k_s = 0.$$

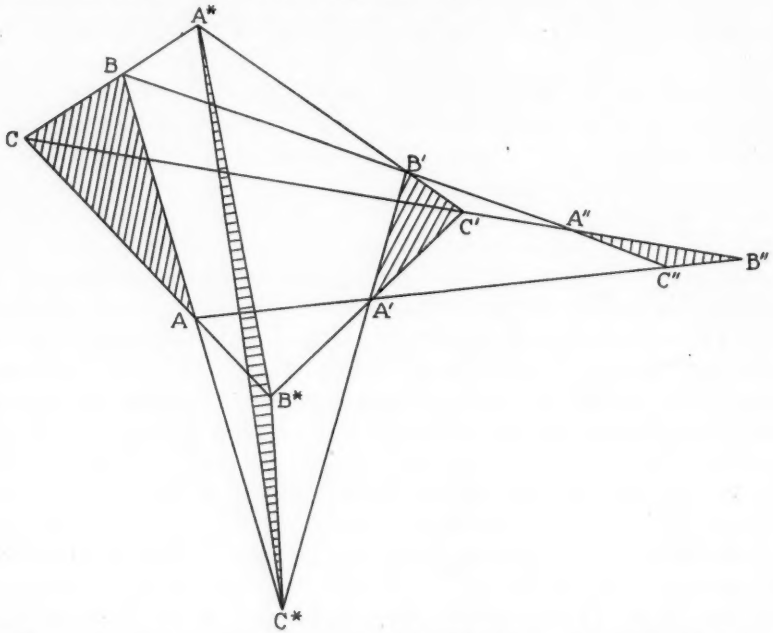
The condition that this cone contain the maximum possible number of mutually perpendicular i spaces, $i = 1, 2, \cdots, n-1$ is that the symmetric function of the ith degree in the $1/k_s$ vanish. This is equivalent to the vanishing of the symmetric functions of the $(n-i)$th degree in the k_s, or C_{i+1} by (55). If $C_1 = 0$, the product of the k_s vanishes, and hence one of them, and by (72) the tangent cone breaks up into two coincident $n-1$ spaces, which checks with the fact that $C_1 = 0$ is the equation of the given locus. We see from the above that $C_{i+1} = 0$ is the locus of points from which the maximum number, $n!/i!(n-i)!$, mutually perpendicular i-spaces can be drawn all tangent to the given quadric.

THE THEOREMS OF CEVA AND MENELAUS AND THEIR EXTENSION[1]

By PAUL WERNICKE, Washington, D. C.

The theorems referred to in the title occur in elementary geometry. That of Ceva states that three concurrent (straight) lines drawn from the vertices A, B, C of a triangle, in its plane, divide the opposite sides BC, CA, AB in ratios the product of which is unity.

Menelaus' theorem, in slightly modernized enunciation, is to the effect that any straight transversal divides the sides BC, CA, AB of a triangle ABC in ratios the product of which is negative unity.



We now consider two triangles, ABC, $A'B'C'$, in a plane S_2, and draw the joins AA', BB', CC' of their vertices. These will in general meet in three points $A''=(BB', CC')$, $B''=(CC', AA')$, $C''=(AA', BB')$. Triangle $A''B''C''$ is the central imperspective triangle of ABC and $A'B'C'$. It reduces to their center of perspective when its area vanishes. The product of the ratios in which the sides BC, CA, AB are divided by the lines AA', BB', CC' will be called the *ceva* of ABC. Similarly for $A'B'C'$.

[1] Read before the Maryland–District of Columbia–Virginia Section of the Association, Dec. 4, 1926.

Finding the intersections $A^*=(BC, B'C'), B^*=(CA, C'A'), C^*=(AB, A'B')$ of pairs of corresponding sides, we also call the product of the ratios in which they divide the sides of either triangle the *menelaus* of the latter. $A^*B^*C^*$ is the axial imperspective triangle of ABC and $A'B'C'$, since it reduces to the so-called axis of perspective when its area vanishes.

We next prove the theorem: *The ceva of ABC is the negative menelaus of $A'B'C'$ (and vice-versa).* In barycentric coordinates with ABC as triangle of reference we write

$$\begin{aligned} A' &= \alpha_0 A + \alpha_1 B + \alpha_2 C \text{ or briefly } = (\alpha_0 A +), \text{ where } && \alpha_0 + \alpha_1 + \alpha_2 \\ B' &= \beta_0 A + \beta_1 B + \beta_2 C \text{ " } \text{ " } = (\beta_0 A +), && = \beta_0 + \beta_1 + \beta_2 \\ C' &= \gamma_0 A + \gamma_1 B + \gamma_2 C \text{ " } \text{ " } = (\gamma_0 A +), && = \gamma_0 + \gamma_1 + \gamma_2 = 1 \end{aligned}$$

Two triangles having the bases CA, AB and a common summit on AA' then have their areas in the ratio α_1/α_2. Accordingly, BC is divided by AA' in the ratio α_2/α_1. The ratios of division in CA and AB are similarly found to be β_0/β_2 and γ_1/γ_0. The ceva of ABC is thus $(\alpha_2\beta_0\gamma_1)/(\alpha_1\beta_2\gamma_0)$.

The ceva and menelaus are dualistically formed products. Each of the former's three factors is a ratio of two barycentric coordinates, this ratio being constant on one of the three joins of vertices, AA', BB', CC'. Each factor of the menelaus is a negative ratio of two perpendiculars dropped from the triangle's vertices onto *any* line drawn through one of the intersections A^*, B^*, C^*. Such a ratio is also that of two barycentric coordinates of such a line and is constant for each of the points A^*, B^*, C^*.

Now BC meets $B'C'$ in $A^*=(1-u)B+uC=(1-v)(\beta_0 A+)+v(\gamma_0 A+)$. The coefficients of A must be equal in the last two members: $(1-v)\beta_0+v\gamma_0=0$. This shows that $v=\beta_0/(\beta_0-\gamma_0)$, $1-v=-\gamma_0/(\beta_0-\gamma_0)$ and that $B'C'$ is divided in the ratio $-\beta_0/\gamma_0$. B^* similarly divides $C'A'$ in the ratio $-\gamma_1/\alpha_1$, and C^* divides $A'B'$ in the ratio $-\alpha_2/\beta_2$. For the menelaus of $A'B'C'$ we find $-(\alpha_2\beta_0\gamma_1)/(\alpha_1\beta_2\gamma_0)$. This proves the theorem.

The vertices of the central imperspective triangles are found to be

$$\begin{aligned} A'' &= (\beta_0\gamma_0 A + \beta_0\gamma_1 B + \beta_2\gamma_0 C) : (\beta_0\gamma_0 + \beta_0\gamma_1 + \beta_2\gamma_0) \\ B'' &= (\gamma_0\alpha_1 A + \gamma_1\alpha_1 B + \gamma_1\alpha_2 C) : (\gamma_0\alpha_1 + \gamma_1\alpha_1 + \gamma_1\alpha_2) \\ C'' &= (\alpha_2\beta_0 A + \alpha_1\beta_2 + \alpha_2\beta_2 C) : (\alpha_2\beta_0 + \alpha_1\beta_2 + \alpha_2\beta_2), \end{aligned}$$

where the denominators have been reduced by means of the relations

$$\alpha_0 + \alpha_1 + \alpha_2 = \beta_0 + \beta_1 + \beta_2 = \gamma_0 + \gamma_1 + \gamma_2 = 1.$$

Its area is proportional to the determinant Δ_1 given below, whose elements are found to be the cofactors of the corresponding elements in the determinant Δ_2, of which it is, therefore, the square.

$$\Delta_1 = \begin{vmatrix} \beta_0\gamma_0 & \beta_0\gamma_1 & \beta_2\gamma_0 \\ \gamma_0\alpha_1 & \gamma_1\alpha_1 & \gamma_1\alpha_2 \\ \alpha_2\beta_0 & \alpha_1\beta_2 & \alpha_2\beta_2 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} 0 & \alpha_2 & -\alpha_1 \\ -\beta_2 & 0 & \beta_0 \\ \gamma_1 & -\gamma_0 & 0 \end{vmatrix}$$

Triangles ABC and $A'B'C'$ are in perspective when this determinant Δ_2 vanishes. This is Ceva's theorem.

The vertices of the axial imperspective triangle being

$$\begin{aligned} A^* &= [\quad (\beta_0\gamma_1)B + (\beta_0\gamma_2)C]:[(\beta_0\gamma_1) + (\beta_0\gamma_2)] \\ B^* &= [(\gamma_1\alpha_0)A \quad + (\gamma_1\alpha_2)C]:[(\gamma_1\alpha_2) + (\gamma_1\alpha_0)] \\ C^* &= [(\alpha_2\beta_0)A + (\alpha_2\beta_1)B \quad]:[(\alpha_2\beta_0) + (\alpha_2\beta_1)], \end{aligned}$$

where $(\beta_i\gamma_j)$ stands for $\beta_i\gamma_j - \beta_j\gamma_i$, its area is proportional to

$$\begin{vmatrix} 0 & (\beta_0\gamma_1) & (\beta_0\gamma_2) \\ (\gamma_1\alpha_0) & 0 & (\gamma_1\alpha_2) \\ (\alpha_2\beta_0) & (\alpha_2\beta_1) & 0 \end{vmatrix} = \begin{vmatrix} 0 & -\gamma_0 & \beta_0 \\ \gamma_1 & 0 & -\alpha_1 \\ -\beta_2 & \alpha_2 & 0 \end{vmatrix} \cdot \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \\ \gamma_0 & \gamma_1 & \gamma_2 \end{vmatrix}.$$

The condition for the vanishing of its area is evidently the same, namely $\Delta_2 = 0$.

If ABC and $A'B'C'$ are in perspective, let the lengths of the perpendiculars from A', B', C' to the axis of perspective $(A^*B^*C^*)$ be a, b, c. Then

$$-\beta_0/\gamma_0 = b/c; \quad -\gamma_1/\alpha_1 = c/a; \quad -\alpha_2/\beta_2 = a/b.$$

Designate the center of perspective by $E = \epsilon_0 A + \epsilon_1 B + \epsilon_2 C = (\epsilon_0 A +)$. Then

$$\alpha_2/\alpha_1 = \epsilon_2/\epsilon_1; \quad \beta_0/\beta_2 = \epsilon_0/\epsilon_2; \quad \gamma_1/\gamma_0 = \epsilon_1/\epsilon_0.$$

We may write

$$\begin{aligned} (1 - u_0)A + u_0(e_0A +) &= (\alpha_0 A +), \\ (1 - u_1)B + u_1(e_0A +) &= (\beta_0 A +), \\ (1 - u_2)C + u_2(e_0A +) &= (\gamma_0 A +), \end{aligned}$$

whence

$$\begin{aligned} u_0 &= \frac{1 - \alpha_0}{1 - e_0}, & \alpha_1 &= e_1 u_0, & \alpha_2 &= e_2 u_0, \\ \beta_0 &= e_0 u_1, & u_1 &= \frac{1 - \beta_1}{1 - e_1}, & \beta_2 &= e_2 u_1, \\ \gamma_0 &= e_0 u_2, & \gamma_1 &= e_1 u_2, & u_2 &= \frac{1 - \gamma_2}{1 - e_2}. \end{aligned}$$

A fourth point of reference, D, is needed if the triangles ABC, $A'B'C'$ lie in a three dimensional space S_3 but no longer in the same plane. The vertices A', B', C' and the center E will then be

$$A' = \alpha_0 A + \alpha_1 B + \alpha_2 C + \alpha_3 D = (\alpha_0 A +), \quad \text{where} \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$$
$$B' = (\beta_0 A +) \qquad = \beta_0 + \beta_1 + \beta_2 + \beta_3$$
$$C' = (\gamma_0 A +) \qquad = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3$$
$$E = (e_0 A +) \qquad = e_0 + e_1 + e_2 + e_3 = 1.$$

The above set of equations then becomes

$$u_0 = \frac{1-\alpha_0}{1-e_0}, \quad \alpha_1 = e_1 u_0, \quad \alpha_2 = e_2 u_0, \quad \alpha_3 = e_3 u_0$$
$$\beta_0 = e_0 u_1, \quad u_1 = \frac{1-\beta_1}{1-e_1}, \quad \beta_2 = e_2 u_1, \quad \beta_3 = e_3 u_1$$
$$\gamma_0 = e_0 u_2, \quad \gamma_1 = e_1 u_1, \quad u_2 = \frac{1-\gamma_2}{1-e_2}, \quad \gamma_3 = e_3 u_2.$$

These give the equations of complanarity of B, C, B', C' and two point-quadruples obtained therefrom by cyclic permutation:

$$(\beta_0\gamma_3) = 0, \quad (\gamma_1\alpha_3) = 0, \quad (\alpha_2\beta_3) = 0 \quad \text{which is} \quad \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = 0$$

and two similar equations indicating complanarity of four points. $B'C'$, $C'A'$, $A'B'$ are again divided in A^*, B^*, C^*, respectively, in the ratios $-\beta_0/\gamma_0$, $-\gamma_1/\alpha_1$, $-\alpha_2/\beta_2$. These by virtue of the conditions of complanarity, may be written $-\beta_3/\gamma_3$, $-\gamma_3/\alpha_3$, $-\alpha_3/\beta_3$. The menelaus of $A'B'C'$ is therefore again -1. The first factor of the ceva, α_2/α_1, is the ratio of the volumes of two tetrahedra whose bases, DBA and DAC, meet in the edge DA and whose common summit is any point of AA'. The interpretation of the two other factors is analogous.

By assuming a fourth point $D' = \delta_0 A + \delta_1 B + \delta_2 C + \delta_3 D$, $\Sigma\delta_i = 1$ on DE, we obtain a tetrahedron $A'B'C'D'$ in perspective with $ABCD$. A fourth row $\delta_0 = \epsilon_0 u_3$, $\delta_1 = \epsilon_1 u_3$, $\delta_2 = \epsilon_2 u_3$, $u_3 = (1-\delta_3)/(1-\epsilon_3)$ is added to our set of equations, leading to three further conditions of complanarity $(\alpha_1\delta_2) = 0$, $(\beta_2\delta_0) = 0$, $(\gamma_0\delta_1) = 0$ and to the relations $\alpha_1\beta_2\gamma_3\delta_0 = \alpha_2\beta_3\gamma_0\delta_1 = \alpha_3\beta_0\gamma_1\delta_2$. They take the place of Ceva's theorem for tetrahedra. From them the corresponding theorems for pairs of corresponding triangular sides are derivable by applying the conditions of complanarity.

The edges of tetrahedron $ABCD$ form three *tetragrams* or quadrilaterals in S_3: $ABCD$, $ACBD$, $ADCB$. Each of these defines the tetrahedron. Ceva's theorem for the tetragram $ABCD$ would read: *Planes laid through a point E and one of the edges AB, BC, CD, DA of a tetragram divide its opposite edges in ratios whose product is unity.*

Pairs of corresponding triangular sides (ABC, $A'B'C'$) of two tetrahedra in perspective, meet in lines containing, each, three intersections of corresponding edges. A complete quadrilateral results, the plane of which may be called, the *base of perspective* of the tetrahedra, or of the tetragrams $ABCD$, $A'B'C'D'$ defining them. Perpendiculars from B', C' to any plane laid through the intersection of ABC with $A'B'C'$ are in the ratio β_0/γ_0, the negative ratio of division of $B'C'$ where it meets BC. Similarly for the remaining edges taken as sides of the triangles ABC, DCB, DAC, DBA, so that each occurs in opposite directions in adjacent triangles.

Menelaus' theorem may then be extended to read: *Any plane meets the sides of a tetragram in points dividing them in ratios the product of which is unity.*

Lastly, we consider two tetragrams $ABCD$, $A'B'C'D'$ in space S_3, but not necessarily in perspective. Lay planes through DA and A', AB and B', etc. These divide the opposite edges BC, CA, etc, in four ratios the product of which may be called the ceva of tetragram $ABCD$. On the other hand, the product of the four ratios in which the planes $a=DCB$, $b=DAC$, $c=DBA$, $d=ABC$ divide the sides $d'a'=B'C'$, $a'b'=C'D'$, $b'c'=D'A'$, $c'd'=A'B'$ would correspondingly be the menelaus of tetragram $A'B'C'D'$. Obviously: *The menelaus of either tetragram equals the other's ceva.* As these products have an even number of factors, there is no difference of sign.

The proof may be indicated by pointing out that BC was found to be divided by plane DAA' in the ratio α_2/α_1, while plane $a=DBC$ divides $b'c'=D'A'$ in the ratio $-\delta_1/\alpha_1$. Both these ratios, each with three similar ones, enter into the expression $(\alpha_2\beta_3\gamma_0\delta_1)/(\alpha_1\beta_2\gamma_3\delta_0)$ for the ceva of $ABCD$ or the menelaus of $A'B'C'D'=a'b'c'd'$.

The theorems here given pave the way for further generalizations, especially in connection with perspective.

REMARKS ON THE PROBABLE ERROR OF A MEAN[1]

By CECIL C. CRAIG, University of Michigan

1. Statement of the problem. The standard deviation of an empirically determined mean based on a sample of N from an infinite parent is well known to be $(\theta_2/N)^{1/2}$ in which θ_2 is the second moment about the mean of the parent distribution. If it be assumed that the distribution of sample means is normal, it is customary to write

$$\theta_1 = m_x \pm .6745(\theta_2/N)^{1/2}$$

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[1] Read before the Michigan Section of the Association, April 28, 1927.

in which θ_1 is the mean of the parent distribution and m_x is the mean of the sample of N. The quantity, $.6745(\theta_2/N)^{1/2}$ is said to be the probable error of the m_x. That is, the probability is $\frac{1}{2}$ that m_x is within the interval $.6745(\theta_2/N)^{1/2}$ measured from θ_1. Or if any other constant k be used in place of .6745, other similar statements may be made; if $k=1$, the probability is 68/100 that m_x is within $(\theta_2/N)^{1/2}$ of θ_1; if $k=3$, the probability is 997/1000 that m_x is within $3(\theta_2/N)^{1/2}$ of θ_1, etc.

The situation is fairly satisfactory for moderately large values of N if the parent even remotely resembles a normal distribution. But one point invites further discussion, namely, that in $(\theta_2/N)^{1/2}$ the $\theta_2^{1/2}$ is itself known only by an empirical determination. It is apparent that the probable error itself has a probable error, which in turn has its probable error, and so on ad infinitum. It is true that we do not know and cannot ever know the true probable error of m_x. Then what can we say?

2. Discussion in case the parent is normal. In case the parent distribution is normal the results needed for a discussion of this question are known. They are,

$$\lambda_1(m_x) = \theta_1\,; \quad \lambda_2(m_x) = \theta_2/N\,;$$

$$\lambda_3(m_x) = \lambda_4(m_x) = \cdots = \lambda_r(m_x) = 0, \qquad r > 2$$

in which $\lambda_r(m_x)$ is the rth semi-invariant of the distribution of sample m_x's. Thus the distribution of sample m_x's is normal. Now also in this case[1]

$$\lambda_1(\sigma_x) = \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\left(\frac{2\theta_2}{N}\right)^{1/2}, \qquad \lambda_2(\sigma_x) = \frac{N-1-2a^2}{N}\theta_2$$

with

$$a = \Gamma\left(\frac{N}{2}\right) \div F\left(\frac{N-1}{2}\right).$$

Now then for the $\theta_2^{1/2}$ in $\theta_1 = m_x \pm k(\theta_2/N)^{1/2}$, what is the best value we can use in terms of our empirically observed quantities? From the equation

$$\lambda_1(\sigma_x) = a(2\theta_2/N)^{1/2}$$

the value of $\theta_2^{1/2}$ is exactly

$$\lambda_1(\sigma_x) \cdot (1/a)(N/2)^{1/2}\,.$$

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[1] Romanowsky, V., *On the moments of the standard deviation*, etc.; Metron, Vol. 5, No. 4 (1925), p. 12.

But what do we have to put in for $\lambda_1(\sigma_x)$? Only our observed σ_x! But this is also an empirically observed quantity; if it is set in for $\lambda_1(\sigma_x)$ it will have to go in with $\pm$ its error expression which is

$$k[(N - 1 - 2a^2)\theta_2/N]^{1/2}$$

and we shall write

$$\theta_1 = m_x \pm \frac{k}{N^{1/2}}\left[\frac{1}{a}\left(\frac{N}{2}\right)^{1/2}\left\{\sigma_x \pm k\frac{(N - 1 - 2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right\}\right]$$

$$= m_x \pm \frac{k}{a\sqrt{2}}\left[\sigma_x \pm k\frac{(N - 1 - 2a^2)^{1/2}}{N}\theta_2^{1/2}\right].$$

But the quantity $\theta_2^{1/2}$ still occurs in the last terms; if we replace it again by

$$\frac{1}{a}\left(\frac{N}{2}\right)^{1/2}\left[\sigma_x \pm k\frac{(N - 1 - 2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right],$$

we have

$$\theta_1 = m_x$$

$$\pm \frac{k}{a\sqrt{2}}\left\{\sigma_x \pm k\frac{(N - 1 - 2a^2)^{1/2}}{N^{1/2}}\left[\frac{1}{a}\left(\frac{N}{2}\right)^{1/2}\left\{\sigma_x \pm k\frac{(N - 1 - 2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right\}\right]\right\}$$

$$= m_x \pm \frac{k}{a\sqrt{2}}\left\{\sigma_x \pm \frac{k}{a\sqrt{2}}(N - 1 - 2a^2)^{1/2}\sigma_x\left[1 \pm k\frac{(N - 1 - 2a^2)^{1/2}}{N^{1/2}}\theta_2^{1/2}\right]\right\}.$$

And repeating this indefinitely, we get,

$$\theta_1 = m_x \pm \frac{k}{a\sqrt{2}}\sigma_x\left[1 \pm \frac{k}{a\sqrt{2}}(N - 1 - 2a^2)^{1/2}\right.$$

$$\left. \pm \left(\frac{k}{a\sqrt{2}}\right)^2(N - 1 - 2a^2) \pm \cdots\right].$$

Suppose all the signs were taken positive. Would the resulting series converge? Obviously this depends on

$$\left|\frac{k}{a\sqrt{2}}(N - 1 - 2a^2)^{1/2}\right| \text{ being less than } 1,$$

with

$$a = \Gamma\left(\frac{N}{2}\right) \div \Gamma\left(\frac{N - 1}{2}\right).$$

The easiest way to answer this question seems to be to remember that

$$a = \frac{\left(\frac{N}{2}\right)^{1/2} \lambda_1(\sigma_x)}{\theta_2^{1/2}},$$

and that

$$(N - 1 - 2a^2)^{1/2} = \frac{N^{1/2}\sigma_\mu}{\theta_2^{1/2}}, \text{ where } \mu = \sigma_x,$$

so that

$$\frac{k}{\sqrt{2}} \frac{(N - 1 - 2a^2)^{1/2}}{a} = k\frac{\sigma_\mu}{\lambda_1(\sigma_x)}, \text{ where } \mu = \sigma_x.$$

Thus for k less than the ratio of *the mean of sample standard deviations* to the *standard deviation of sample standard deviations*, the series does converge and to the value,

$$\theta_1 = m_x + \frac{k\sigma_x}{a\sqrt{2}} \frac{1}{1 - k\frac{\sigma_\mu}{\lambda_1(\sigma_x)}}, \text{ where } \mu = \sigma_x.$$

This can be approximately evaluated for particular values of k by use of the approximate expressions

$$\lambda_1(\sigma_x) = \theta_2^{1/2}\left(1 - \frac{3}{4N} - \frac{7}{32N^2} - \frac{9}{128N^3} - \cdots\right)$$

$$\sigma_\mu = \left(\frac{\theta_2}{2N}\right)^{1/2}\left(1 - \frac{1}{4N} - \frac{3}{8N^2} - \cdots\right)^{1/2}, \text{ where } \mu = \sigma_x,$$

which I have also obtained independently of Romanowsky (loc. cit.).

3. Interpretation of the results. But now what is the meaning of such an expression as the one given just above? The answer is, first of all, that it does not mean so much as one is tempted to think at first glance. It is as true as ever that no formula can give us a better guess at θ_1 than m_x, or a better estimate of $\theta_2^{1/2}$ than $\sigma_x/(1-3/(4N)-7/(32N^2)-\cdots)$. Then what do we have? The answer to this seems to be reached thru considerations like this: Our expression $\theta_1 = m_x \pm .6745(\theta_2/N)^{1/2}$ is a good one for θ_1 if the probable error of m_x is small compared with m_x itself. And our estimate of the value of $\theta_2^{1/2}$ to be used in the formula is satisfactory if its probable error is small compared with itself. And if, further, the like can be said of the probable error of this new probable error and so on indefinitely, the whole situation is what is desired. But if this is not true, if in particular the series set up above had not converged, our probable error of m_x and the value of θ_1 become meaningless. Of course not only con-

vergence is desired but rapid convergence. However for even moderate values of N this is effected for the values of k used in practice in the case we have considered.

Of course it was to be expected that the result of the investigation in this one of the most favorable of cases would be satisfactory. But it is important to emphasize that by so turning out it has added nothing to the expression for θ_1 or to the expression for the probable error of m_x beyond the usual meaning given to them. If this investigation or any like it should turn out unfavorably it would be too bad. And the difficulties to be encountered in most other cases do not lessen the necessity of knowing that such an investigation in each case would turn out favorably.

AN ELEMENTARY EXAMPLE OF A CONTINUOUS NON-DIFFERENTIABLE FUNCTION

By FRED W. PERKINS, Harvard University

The first example showing that the continuity of a function of the real variable x does not imply the existence of the derivative of this function at a single point is due to Weierstrass.[1] Other authors have also written upon this subject, but, in view of the importance of the phenomenon, the following discussion, which the writer has endeavored to make as elementary as possible, may not be without interest.

The method by which we propose to construct the function $f(x)$ constituting our example depends upon an interpolation process by means of which we extend the definition of $f(x)$ after we have defined this function for certain pairs of values of x. Let $x=x'$ and $x=x''$ be such a pair of values $(x' \neq x'')$ and denote by y' and y'' the numbers $f(x')$ and $f(x'')$ respectively. We set

$$f(x' + \tfrac{1}{3}[x'' - x']) = y' + \tfrac{5}{6}[y'' - y']$$

and

$$f(x' + \tfrac{2}{3}[x'' - x']) = y' + \tfrac{1}{6}[y'' - y'].$$

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[1] Weierstrass gave an example in lectures at Berlin as early as 1861, but it was not published until 1875. Weierstrass' discussion may be found in his *Werke* (Berlin 1895) Vol. 2, page 71ff. Wiener (Journal für Mathematik vol. 90, 1881, p. 221ff.) made a study of the function constituting this example; in connection with this paper, see a note by Weierstrass, (loc. cit. vol. 2, page 228ff.).

Riemann gave a function which enables us to infer that a continuous function may fail to have a derivative on an everywhere dense set of points. (*Gesammelte Mathematische Werke*, p. 225ff.). See also Schwartz, *Gesammelte Mathematische Abhandlungen* (Berlin, 1890), vol. 2, p. 269ff. Among the more recent contributions on this subject may be mentioned two examples of nowhere differentiable functions published in the Jahresbericht der Deutschen Mathematiker Vereinigung, by Faber, vol. 16 (1907), p. 538ff. and by Landsberg, vol. 17 (1908), and a memoir on the Weierstrass' function by Hardy in the Transac-

It will be convenient to represent by P' and P'' the points (x', y') and (x'', y'') respectively, and also to denote by P_1 the point (x_1, y_1) and by P_2 the point (x_2, y_2). Excluding the case that $y' = y''$, we see that y_1 and y_2 lie between y' and y'', and that each of the quantities $|y_1 - y'|$, $|y_2 - y_1|$, $|y'' - y_2|$ is positive, but not greater than $5|y'' - y'|/6$. Furthermore, the lines $P'P_1$, P_1P_2 and P_2P'' have slopes which in each case are at least twice as great, numerically, as the slope of $P'P''$.

We are now ready to define the function $f(x)$ on the interval $0 \leqq x \leqq 1$. We set $f(0) = 0$ and $f(1) = 1$, and use the process described above to determine $f(1/3)$ and $f(2/3)$. We shall call the two points thus obtained the "interpolated points of the first order." By the same method we interpolate two more points between each pair of adjacent points now known on the graph of $f(x)$. The six points thus determined we shall call the "interpolated points of the second order," and so on, indefinitely. In this way, the value of $f(x)$ is specified for all values of x of the form $p/3^n$, where n is a positive integer, and p is a non-negative integer, not greater than 3^n.

Let X be any number of the interval $0 \leqq x \leqq 1$ not of the form $p/3^n$. Since $f(x)$ is defined for values of x differing from X by less than any preassigned positive quantity, and since the function $f(x)$, insofar as it has already been defined, has the lower bound 0 and the upper bound 1, the set of all known points on the graph of $f(x)$ has at least one limit point on the line-segment determined by the relations $x = X$, $0 \leqq y \leqq 1$. We shall now show that there is in fact just one such limit point. From properties of the interpolation process already stated, it follows that for any positive integer n, and any non-negative integer p such that $p < 3^n$ we have

$$\left| f\left(\frac{p+1}{3^n}\right) - f\left(\frac{p}{3^n}\right) \right| \leqq \left(\frac{5}{6}\right)^n.$$

Any interpolated point with an abscissa lying between $p/3^n$ and $(p+1)/3^n$ has an ordinate lying between $f[p/3^n]$ and $f[(p+1)/3^n]$. Given any $\epsilon > 0$, we can choose n so large that $(5/6)^n < \epsilon$, and then choose p so that X is an interior point of the interval bounded by $p/3^n$ and $(p+1)/3^n$. Then, in so far as $f(x)$ is defined on this interval, its least upper and greatest lower bounds here differ by less than ϵ, which shows that the set of all known points on the graph of $f(x)$ cannot have more than one limit point with the abscissa X. Let $f(X)$ be defined as the ordinates of this unique limit point. We extend in this manner the definition of $f(x)$ throughout the entire unit interval. The resulting function is clearly continuous at each of the non-interpolated points.

tions of the American Mathematical Society, vol. 17 (1916), p. 301. See also Hobson, The Theory of Functions of a Real Variable, (Cambridge, University Press, 1907), p. 620ff. In the paper by Hardy will be found further references to the literature on this subject.

Further, if $x^* = p/3^n$ is the abscissa of an interpolated point each of the two adjacent interpolated points of the $(n+k)$-th order has an ordinate differing from $f(x^*)$ by not more than $(5/6)^{n+k}$. By choosing the positive integer k sufficiently large, this difference can be made less, in absolute value, than any preassigned positive ϵ. It follows that for such a choice of k, the relation $|f(x^*) - f(x)| \leqq \epsilon$ holds for all interpolated points of the interval

$$\frac{3^k p - 1}{3^{n+k}} \leqq x \leqq \frac{3^k p + 1}{3^{n+k}},$$

and so at their limit points as well. Hence, $f(x)$ is continuous at the interpolated points also. The same sort of reasoning shows that $f(x)$ is continuous at $x = 0$ and $x = 1$.

We are now ready to show that at no point of the interval $0 \leqq x \leqq 1$ does the continuous function $f(x)$ possess a derivative. We understand this to mean that for no value of x in the interval in question does the difference quotient $[f(x+h) - f(x)]/h$ approach a limit as h approaches zero.[1]

Let x_0 be any number of the interval $0 \leqq x \leqq 1$. We choose any positive integer n, and then determine the positive integer p so that $(p-1)/3^n \leqq x_0 \leqq p/3^n$. One of the points

$$R: \left[\frac{p-1}{3^n},\ f\left(\frac{p-1}{3^n}\right)\right], \qquad S: \left[\frac{p}{3^n},\ f\left(\frac{p}{3^n}\right)\right]$$

determines with the point P_0: $[x_0, f(x_0)]$ a line with slope at least as great numerically as that of the line RS. But, from a property of the interpolation process, we know that the absolute value of the slope of RS is at least 2^n. Hence, we see that by choosing n sufficiently large we can find a point on the curve as near as we like to P determining with P a line with slope greater numerically than any preassigned quantity. This proves that $f(x)$ has no derivative for $x = x_0$. It does not imply, of course, that the difference quotient $[f(x_0+h) - f(x_0)]/h$ becomes infinite independently of the manner in which h approaches zero. It may be noted, however, that for $x_0 = 0$ or $x_0 = 1$, the difference quotient does become positively infinite independently of the manner in which the point $x_0 + h$ of the unit interval approaches x_0.

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[1] Some writers, on the contrary, use a definition of differentiability allowing the difference quotient either to approach a limit or to become positively or negatively infinite.

PI MU EPSILON MATHEMATICAL FRATERNITY

By TOMLINSON FORT, Lehigh University

As a college teacher who has found the Pi Mu Epsilon fraternity of great benefit to his work and in the belief that the organization has potentialities for a widespread benefit to American mathematics, I presume to tell the readers of the MONTHLY something about it.

The fraternity was founded at Syracuse in 1914 by Professor E. D. Roe, Jr. and incorporated under the laws of the State of New York on May 25 of that year. Its headquarters have remained at Syracuse and Professor Roe has remained its Director General or chief officer. His deep interest in the fraternity and his wise handling of its affairs constitute no small contribution to the cause of mathematics and scholarship.

Pi Mu Epsilon is not a secret fraternity nor a fraternity based on anything but interest and achievement in mathematics. It is not an undergraduate mathematics club but numbers among its members undergraduates, graduates, and faculty members on a plane of equality within the fraternity. It aims to combine with the ideals of the more widely known Phi Beta Kappa a working organization within the colleges and universities for study and for the promotion of all mathematical interests within the community. It undertakes to rival no organization. It has concentrated on work at the institution and its great good is unquestionably to inspire enthusiasm in the student and to give him stimulating associations and an opportunity for independent study. There could be no greater mistake than to feel that Pi Mu Epsilon is or ever should be, in thought or deed or appearance, a rival of the American Mathematical Society or of the Mathematical Association of America. It works at the universities bringing the faculty, graduates, and leading undergraduates together for the common cause.

Why Pi Mu Epsilon rather than local clubs only? It is a matter of results. We must meet American college life as it is. Those of us who have tried both, find that Pi Mu Epsilon succeeds the better. We are able to set up strict scholarship qualifications and the student works for the honor of membership and of wearing the key. A campus reputation and recognition is quickly obtained for Pi Mu Epsilon and it becomes second to but few college honors. The student understands the national organization. The very fact that it is national makes him rate it higher. The fact that it is called "fraternity" tends to break down the barrier between him and his teachers quicker than anything else that I know. My own association with my students in Pi Mu Epsilon has been one of the keenest pleasures of my experience as a teacher.

What do the chapters of Pi Mu Epsilon do? There is considerable variation. Each chapter has its own by-laws but is subject to the national constitution.

Usually no one is admitted who has not completed at least two semesters of calculus or who has a grade less than "B" in any mathematics course. He must in addition be reported on by a scholarship committee as to scholarship in all his work and enthusiasm and probable success in mathematics. Some chapters conduct their work much as is usual in mathematics clubs with added social features. I, of course, can speak particularly only of my own experience. At Hunter College Pi Mu Epsilon, in addition to being an award for merit, has been primarily a study organization conducted at times somewhat as a seminar and at times in a way that might better be described as a socialized recitation. We have universally given all of one semester to a single topic. These topics have been: Non-Euclidean geometry, relativity, statistics, history of calculus, and vector analysis. Everybody prepares. We have had delightful social gatherings in addition to the study hours. At Hunter College there is a general mathematics club open to all, having many members who would not be eligible for Pi Mu Epsilon. The Pi Mu Epsilon members are all members of the "Mathematics Club." We regard its support as a duty incumbent on every member of the fraternity. We have also a mathematics club with membership limited to freshmen and programs of corresponding advancement. Pi Mu Epsilon members undertake to organize and assist the freshman club. Our chapter also awards each year a prize of ten dollars to the student graduating with the highest grades in mathematics from Hunter High School which is under different management from the College and is in a different part of the city. A condition of the awarding of the prize is that an undergraduate member of Pi Mu Epsilon be permitted to award it in the name of the fraternity from the High School commencement stage. We hope that this prize has done something for mathematics in the high school. Pi Mu Epsilon undergraduate members recently were instrumental in organizing a mathematics club at Washington Irving High School, New York City. At the University of Alabama, Pi Mu Epsilon members, in addition to making arrangements for their own meetings, were instrumental each year in organizing the Newtonian (freshman mathematics) Club, sponsored some outside lectures, and tried in various other ways to stimulate interest in mathematics on the campus.

What is necessary for a successful chapter of Pi Mu Epsilon? Every organization derives its spirit and owes its success to a few enthusiastic persons. Certainly one and preferably several members of the faculty who in age, position, and personality command the respect of every one and who will enter into Pi Mu Epsilon whole-heartedly and genuinely are necessary for its success. A department where research is the whole thing, teaching a necessary drudgery and undergraduate students regarded without true respect will never do. This is not to imply, however, that the fraternity does not recognize the importance of research and hope to lead the best of its young members to become research

mathematicians. To turn Pi Mu Epsilon over to a young instructor as director and to have it neglected by the older members of the staff will cause it to be a languishing mathematics club. One might just as well have it such in name. Pi Mu Epsilon will go but, of course, the enthusiastic teacher must lead. He will be repaid, I am sure, in those things which a true teacher prizes. A weak department without scholarship will not be granted a charter. The chapters of the fraternity decide this matter, each chapter casting one vote, but the spirit of the fraternity now is to vote favorably only on chapters where there is assurance of a scholarly faculty and of enthusiasm for the fraternity. No charter has ever been voted to an organization of students only.

The chapters of Pi Mu Epsilon are: Syracuse University, Ohio State University, University of Pennsylvania, University of Missouri, University of Alabama, Iowa State College, University of Illinois, Bucknell University, University of Montana, Hunter College of the City of New York, Washington University, University of California at Los Angeles, University of Kentucky, Ohio Wesleyan University.

The general officers are: E. D. Roe, Jr., Syracuse University, Director General; Tomlinson Fort, Hunter College, Vice-Director General; H. S. Everett, Bucknell University, Secretary General; I. S. Carroll, Syracuse University, Deputy Secretary General; Louisa M. Lotz, University of Pennsylvania, Treasurer General; Mabel Kessler, University of Pennsylvania Librarian General; R. D. Carmichael, University of Illinois; E. R. Hedrick, University of California at Los Angeles; S. E. Rasor, Ohio State University; W. H. Roever, Washington University, Members of Executive Committee General.

QUESTIONS AND DISCUSSIONS

Edited by H. E. Buchanan, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. Line Values of Powers of Trigonometric Functions and Their Use in Constructing Curves[1]

By Dr. H. A. Bender, University of Illinois

The line values of the different powers of the trigonometric functions readily lend themselves to the graphical construction of certain curves, and from the

[1] Read before the Illinois Section of the Association, May 9, 1925, under a different title.

simple geometric construction used to locate one point of the curve one can, in most cases, determine the exact graph by perceiving the locus of the point constructed as the angle be allowed to vary. For example, if at a point a units from the pole on the initial line a perpendicular be drawn to the radius vector, then the foot of the perpendicular is a point of the curve $\rho = a \cos \theta$. The graph, which is the locus of the point constructed as the angle be allowed to vary, is at once apparent from the construction.

The most familiar illustration of this type of graphing of Cartesian equations is afforded by the eccentric angle construction of a point of an ellipse from its parametric equations. Similar construction of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ from its parametric equations $x = a \sec \theta$, $y = b \tan \theta$ is not so familiar but is given in some of the text-books.

In the more complex curves, if one can not determine the exact shape of the curve from a single construction, a second or third point may be located with ease. In case accuracy is desired, one can locate several points with considerable ease and accuracy and hence avoid the computing and plotting a table of points.

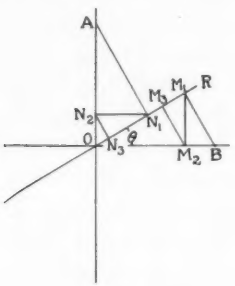

FIG. 1

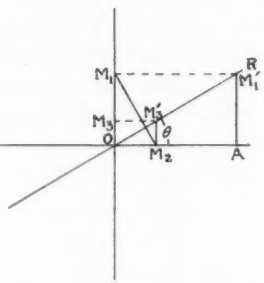

FIG. 2

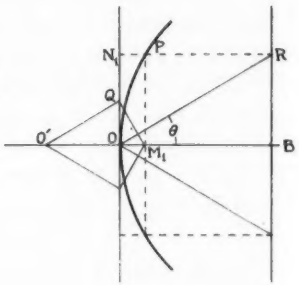

FIG. 3

The line values of $a \sin^n \theta$, $a \cos^n \theta$, and $a \tan^n \theta$ can readily be constructed. In Fig. 1 let OA be equal to a, and from A draw a perpendicular to OR; then $ON_1 = a \sin \theta$, $ON_2 = a \sin^2 \theta$, $ON_3 = a \sin^3 \theta$, etc. Similarly if $OB = a$, then $OM_1 = a \cos \theta$, $OM_2 = a \cos^2 \theta$, $OM_3 = a \cos^3 \theta$, etc. In Fig. 2 at A $(a,0)$ erect a perpendicular and let M_1 be the projection of M_1' on the y-axis, and from M_1 draw a perpendicular to OR; then we have $OA = a$, $OM_1 = a \tan \theta$, $OM_2 = a \tan^2 \theta$, $OM_3 = a \tan^3 \theta$, etc. The reciprocals of these functions may be constructed in the reverse order, or we may in Fig. 1 suppose $ON_3 = a$, $ON_2 = a \csc \theta$, $ON_1 = a \csc^2 \theta$, $OA = a \csc^3 \theta$, etc. Similarly for $a \sec^n \theta$ and $a \cot^n \theta$. The parabola $y^2 = 4ax$ has the parametric equations $x = a(\sec^2 \theta - 1)$, $y = 2a \tan \theta$. In Fig. 3 draw the line $x = 2a$. At O' $(-a,0)$construct an angle equal to θ, then $O'Q = a \sec \theta$ and $O'M_1 = a \sec^2 \theta$, hence $x = OM_1 = a(\sec^2 \theta - 1)$ and $y = ON_1 = 2a \tan \theta$.

The astroid $x^{2/3}+y^{2/3}=a^{2/3}$ has the parametric equations $x=a\cos^3\theta$, $y=a\sin^3\theta$; and from the construction in Fig. 4, $x=OM_3=a\cos^3\theta$, and $y=ON_3=a\sin^3\theta$. The curve $(x/a)^2+(y/b)^{2/3}=1$ has the parametric equations $x=a\cos\theta$,

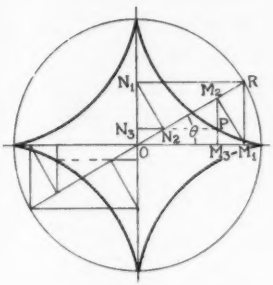

FIG. 4

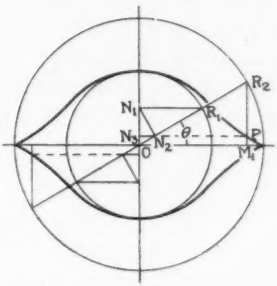

FIG. 5

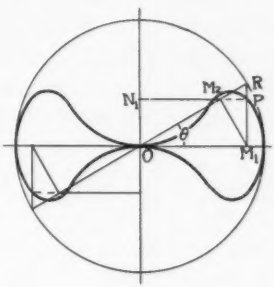

FIG. 6

$y=b\sin^3\theta$; and in Fig. 5, $x=OM_1=a\cos\theta$, $y=ON_3=b\sin^3\theta$. The curve $a^4y^2=a^2x^4-x^6$ has the parametric equations $x=a\cos\theta$, $y=a\cos^2\theta\sin\theta$. In Fig. 6, $x=OM_1=a\cos\theta$, and $OM_2=a\cos^2\theta$, $y=ON_1=a\cos^2\theta\sin\theta$.

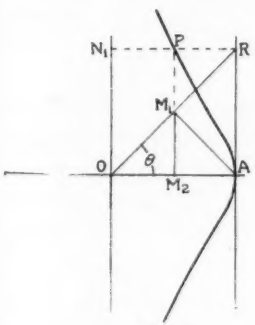

FIG. 7

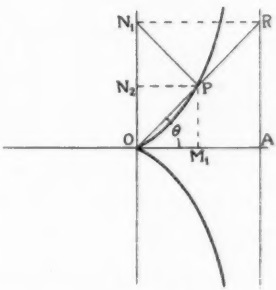

FIG. 8

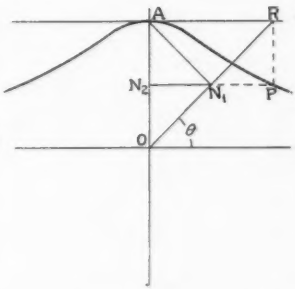

FIG. 9

We might extend the application to the curve $xy^2=4a^2(2a-x)$ as given in Fig. 7. It has the parametric equations $x=2a\cos^2\theta$, $y=2a\tan\theta$. If $OA=2a$, then $x=OM_2=2a\cos^2\theta$ and $y=ON_1=2a\tan\theta$. The cissoid $y^2(2a-x)=x^3$ in Fig. 8 has the parametric equations $x=2a\sin^2\theta$, $y=2a\tan\theta\sin^2\theta$. If $OA=2a$, then $ON_1=2a\tan\theta$, $OP=2a\tan\theta\sin\theta$, $x=OM_1=2a\tan\theta\sin\theta\cos\theta=2a\sin^2\theta$, and $y=ON_2=2a\tan\theta\sin^2\theta$. Since the witch $x^2y=4a^2(2a-y)$ has the parametric equations $x=2a\cot\theta$, $y=2a\sin^2\theta$, its construction in Fig. 9 is at once apparent.

Similar construction is applicable to curves in polar coordinates. The cardioid $\rho=a(1-\cos\theta)$ may be constructed by drawing, in Fig. 10, the circle

O' with radius a. At O' construct an angle equal to θ. Then $O'M = a\cos\theta$, and $OM = OP = \rho = a(1-\cos\theta)$. To construct the curve $\rho = a\sin 2\theta$, let $OA = 2a$ in Fig. 11; then $OM_1 = 2a\sin\theta$, and $OM_2 = OP = \rho = 2a\sin\theta\cos\theta$ (note that ρ is negative for those values of θ for which M_2 is to the left of the origin). The curve $\rho = a\tan(\theta/2)$ has a very simple geometric construction. In Fig. 12, if OR' bisects the angle θ, then if from $A(0,a)$ a perpendicular be drawn to OR', the point P at the intersection of this perpendicular with OR is a point on the curve, for $OM = a\sin(\theta/2)$ and $OP = \rho = a\sin(\theta/2)\sec(\theta/2) = a\tan(\theta/2)$.

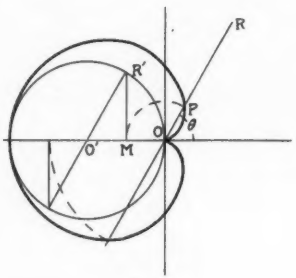

FIG. 10

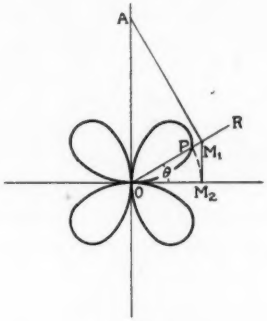

FIG. 11

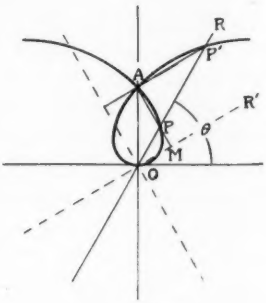

FIG. 12

It follows from the illustrations above that this method is applicable to practically all the standard curves, as well as many of the problems, which are given in our average analytic geometry and calculus text-books. However, from the periodic nature of the trigonometric functions it is evident that the method is best applicable to those curves which are symmetrical with respect to one or both of the coordinate axes.

II. An Example in Probability

By R. E. Morris, University of Montana

It has been shown that the probability of winning in the case of two people A and B, A having 3 coins, B having 2 coins, A winning when he throws more heads than B and B winning when he throws the same number of heads as A or more than A, is $1/2$. This is true for 4 coins and 3 coins, or 5 and 4 coins, or in fact for n and $n-1$ coins. The purpose of this is to show that the probability is $1/2$ and in the course of this procedure to bring to light some interesting relations among binomial coefficients. The expression for the probability of A's winning is

$$\frac{1}{2^n}+\frac{{}_nC_1}{2^n}\,\frac{2^{n-1}-1}{2^{n-1}}+\frac{{}_nC_2}{2^n}\,\frac{[(2^{n-1}-1)-{}_{n-1}C_1]}{2^{n-1}}$$
$$+\frac{{}_nC_3}{2^n}\,\frac{[(2^n-1)-({}_{n-1}C_1+{}_{n-1}C_2)]}{2^{n-1}}+\cdots$$
$$+\frac{{}_nC_{n-1}}{2^n}\,\frac{[(2^{n-1}-1)-({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})]}{2^{n-1}}.$$

Factoring $2^{n-1}-1$, we obtain

$$\frac{1}{2^n}+\frac{1}{2^n2^{n-1}}\{(2^{n-1}-1)({}_nC_1+{}_nC_2+\cdots+{}_nC_{n-1})-[{}_nC_2\,{}_{n-1}C_1\cdot$$
$$+{}_nC_3({}_{n-1}C_1+{}_{n-1}C_2)+\cdots+{}_nC_{n-1}({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})]\}.$$

Examination shows us that the theorem will be proved if we show that this expression is equal to

$$\frac{1}{2^n}+\frac{1}{2^n2^{n-1}}[(2^{n-1}-1)(2^n-2)-(2^{n-1}-1)(2^n-2-2^{n-1}])$$

plus an expression which clearly reduces to $1/2$.
The problem resolves itself into showing that:

$$(1)\qquad (2^{n-1}-1)(2^n-2-2^{n-1})\equiv {}_nC_2\,{}_{n-1}C_1+{}_nC_3({}_{n-1}C_1+{}_{n-1}C_2)+\cdots$$
$$+{}_nC_{n-1}({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})].$$

Clearly $2^{n-1}-1\equiv{}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1}$ and $2^n-2\equiv{}_nC_1+{}_nC_2+\cdots+{}_nC_{n-1}$. Hence

$$(2)\qquad (2^{n-1}-1)(2^n-2-2^{n-1})\equiv({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})[{}_nC_1+{}_nC_2+$$
$$\cdots+{}_nC_{n-1}-(1+{}_{n-1}C_1+\cdots+{}_{n-1}C_{n-1})].$$

If we perform the multiplication indicated in (2) and omit the members contained in (1) we obtain

$${}_{n-1}C_1({}_nC_1-1-{}_{n-1}C_1\,{}_{n-1}C_2-\cdots-{}_{n-1}C_{n-1})+{}_{n-1}C_2({}_nC_1+{}_nC_2-1$$
$$-{}_{n-1}C_1-{}_{n-1}C_2-\cdots-{}_{n-1}C_{n-1})+\cdots+{}_{n-1}C_{n-1}({}_nC_1+{}_nC_2+\cdots$$
$$+{}_nC_{n-1}-1-{}_{n-1}C_1-{}_{n-1}C_2-\cdots-{}_{n-1}C_{n-1}).$$

If we show this last expression to be identical with zero, (1) will follow immediately.
To show that it is equal to zero, let us write it in the form

$$(3)\qquad {}_nC_1({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})+{}_nC_2({}_{n-1}C_2+{}_{n-1}C_3+\cdots$$
$$+{}_{n-1}C_{n-1})+\cdots+{}_nC_{n-1}\,{}_{n-1}C_{n-1}-(1+{}_{n-1}C_1+{}_{n-1}C_2+\cdots$$
$$+{}_{n-1}C_{n-1})({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1}).$$

Rewriting the negative expression, changing order of multipliers, (3) becomes

$$(4)\qquad \begin{aligned}&(1+{}_{n-1}C_1)({}_{n-1}C_1+{}_{n-1}C_2+\cdots+{}_{n-1}C_{n-1})+({}_{n-1}C_1+{}_{n-1}C_2)({}_{n-1}C_2\\&\quad+{}_{n-1}C_3+\cdots+{}_{n-1}C_{n-1})+\cdots+({}_{n-1}C_{n-2}+{}_{n-1}C_{n-1})({}_{n-1}C_{n-1}).\end{aligned}$$

If we make the substitution ${}_nC_r\equiv{}_{n-1}C_r+{}_{n-1}C_{r-1}$ for the left-hand factors in (4) we obtain an expression from which (3) is shown immediately to be zero.

Hence (1) is true and we obtain 1/2 as the probability that either A or B will win. There are some interesting possibilities to this problem with n and $n-2$ coins, n and $n-3$ coins, etc.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

NEW BOOKS RECEIVED

BENNEY, L. B. *Mathematics for Students of Technology*. Oxford, University Press, 1927. vii+451 pages.

BIGGS, H. F. *Wave Mechanics*. Oxford, University Press, 1927. 77 pages. $1.50.

CAREY, F. S. and GRACE, S. F. *Four-Place Tables with Forced Decimals*. London, Longmans, Green, and Company, 1927. 39 pages. $0.40.

DOWSETT, J. F. *Advanced Constructive Geometry*. Oxford, University Press, 1927. viii+340 pages. $9.00.

GIBBS, R. W. M. *Algebra to the Quadratic*. Oxford, University Press, 1927. 160 pages. $1.00.

GODFREY, C. and SIDDONS, A. W. *Four-Figure Tables*. Cambridge, University Press, 1927. 40 pages.

HOBSON, E. W. *Theory of Functions of a Real Variable and the Theory of Fourier's Series*. Volume I, third edition. Cambridge, University Press, 1927. xv+736 pages.

LITTLE, A. S. *A Table of Interpolation Multipliers*. Boston, Financial Publishing Company, 1927. 29 pages.

LOVE, C. E. *Analytic Geometry*. Revised edition. New York, The Macmillan Company, 1927. xiv+257 pages.

MASON, T. E. and HAZARD, C. T. *Analytic Geometry*. Boston, Ginn and Company, 1927. xi+224 pages. $2.40.

OTIS, A. S. and WOOD, B. D. *Columbia Research Bureau Algebra Test*. Yonkers-on-Hudson, World Book Company, 1927.

PERRON, O. *Algebra*. I. *Die Grundlagen*; II. *Theorie der algebraischen Gleichungen*. Berlin, Walter de Gruyter and Company, 1927. viii+307 pages and viii+243 pages.

SMITH, D. E. and REEVE, W. D. *The Teaching of Junior High School Mathematics.* Boston, Ginn and Company, 1927. viii+411 pages. $2.00.

WEIDA, F. M. *The Logarithmic Slide-Rule.* New York, The Macmillan Company, 1927. 7 pages.

REVIEWS

Cremona Transformations in Plane and Space. By HILDA P. HUDSON. Cambridge, England, University Press, 1927. xx+454 pages. Price £2 2s.

The Cambridge University Press is famous for its comprehensive list of standard works on fundamental subjects of mathematics. Geometers will welcome this one to a worthy place in that distinguished company. The general plan is an elevated one; in the opening chapter the reader is introduced to homaloidal nets, postulation and equivalence of plane systems, the direct and inverse fundamental systems, followed in the second by Clebsch's theorem and its arithmetic consequences. Quadratic transformations are discussed under three headings; planes distinct, planes superposed, involutions, in 24 pages, yet nearly every known result is included. Then follows the discussion of series of composition, involution, and the application to the resolution of singularities of plane curves. This last chapter could have been made more useful and attractive by the use of figures and of more illustrative examples. The theory of plane transformations is well rounded out in the first third of the book.

In space, one immediately feels a different atmosphere—here so much remains still undone, and the road is by no means clear how to proceed further. Postulation and equivalence are treated at length, but the various genera of a surface are hardly mentioned. In the discussion of particular elements the concept of contact has a prominent position; it is pointed out that not only do the ordinary formulas fail in this case, but that higher singularities involving contact cannot be expressed in terms of simpler ones linearly. Then follows a detailed discussion of a number of special transformations of cubics, most of which is the work of the author. The minute study of a (3, 4) transformation defined by cubics with a basic conic and osculating a line is a gem. A chapter is devoted to the resolution of singularities of surfaces, following the methods of Levi and Chisini. It represents a great amount of careful work, yet the presentation is sketchy and frequently not convincing. There is still plenty of room for the efforts of the most skillful investigators in this field.

At the end of the book the characteristics of the plane Cremona transformations of orders 2 to 16 are given, and the 75 forms of cubic transformations in space. The book contains no foot-notes. At the end appear the titles, names of authors, and complete reference to every article made use of in the book. In

the text, usually in a paragraph heading, appears a number in type used for no other purpose, referring to a specific entry in the list at the end, while after each paper appear the numbers of the pages in which that article is cited. The list is not complete, but is nearly enough so to make the work a veritable hand-book. Indeed the author states explicitly that the list is incomplete, and generously leaves a few blank pages for further entries. Typographically the work has attained a degree of excellence hardly found elsewhere; errors in the numerous formulas do not exist. It is particularly appropriate that an author who has enriched the field by so many important contributions has now put them in a proper setting by presenting this well-proportioned and carefully elaborated treatise on the whole subject.

VIRGIL SNYDER

Mathematical Statistics. By H. L. RIETZ. The Carus Mathematical Monographs. Number Three. Chicago, The Open Court Publishing Company, 1927. xi+181 pages.

This little book serves as an admirable introduction to Mathematical Statistics. Although the mathematics involved does not go beyond the calculus, and in most chapters not often beyond college algebra, nevertheless, very fundamental concepts in the theory of statistics and probability are set forth in unusually clear form. This is no easy task to perform and the ability to carry it through successfully comes only from long experience in dealing with practical problems in statistics and familiarity with the literature on the subject. It is difficult enough to get these concepts straightened out in one's own mind not to mention getting them across to others, particularly if they are not familiar with the elements of statistics. This monograph is a striking illustration of the fairly successful carrying out of the idea and purpose of the Carus Foundation. There are seven short chapters dealing with the following topics: the underlying concepts of mathematical statistics, relative frequencies in simple sampling, frequency functions of one variable, correlation, random sampling fluctuations, the Lexis theory, and a development of the Gram-Charlier series. The author has not attempted a detailed exposition, but he has emphasized and brought into relief the high spots under these captions.

In the first chapter stress is laid on two general types of problems which occur in mathematical statistics. In connection with problems falling under these two types the author gives detailed consideration to certain underlying concepts, taken in pairs, as follows: relative frequency and probability, observed and theoretical frequency distributions, arithmetic mean and mathematical expectation, mode and most probable value, moments and mathematical expectations of a power of a variable. The second chapter brings out the important points in

connection with the binomial description of frequency together with the mathematical expectation and standard deviation of the number of successes. The theorem of Bernoulli is explained and the proof by the use of the Bienaymé-Tchebycheff criterion pointed out. Some attention is also given to the DeMoivre-Laplace theorem and the Poisson exponential function.

A great deal of matter is compressed in the chapter of thirty pages on frequency functions of one variable. The Pearson frequency curves and the Gram-Charlier series of Type A and Type B are considered together with some remarks on determining the coefficients and on skewness and excess. Simple, multiple, and partial correlation are treated in the next chapter both with regard to the regression method and the correlation surface method of description. The treatment of multiple and partial correlation undoubtedly suffers on account of the condensation. A very good explanation is given of the correlation ratio in non-linear regression. The important subject of random sampling fluctuations is unusually well set forth in view of the limited space at the author's disposal. The reviewer would regard this as about the best chapter in the book.

Urn schemata are employed in the next chapter to illustrate the Bernoulli, Poisson, and Lexis types of statistical series arising in practice. The usual formulas for the mathematical expectation and standard deviation are obtained for these types with some illustrations of subnormal, normal and supernormal dispersion as obtained by the Lexis ratio. This last chapter contains a development of Type A and Type B Gram-Charlier series. The values of the coefficients in the Type A series are obtained by the use of the biorthogonal property of the normal function and its derivatives and the Hermite polynomials. The rigid and particular form of the analysis, due to Wicksell, is very satisfactory because in most texts dealing with this subject the proofs are sloppy. It is to be regretted that the author could not devote a chapter to semi-invariants and explain their elegant and useful properties in connection with the series development of frequency distributions. The value of these developments is their obvious extension to distributions of two or more variates.

The author concludes with several pages of notes giving references to important sources of material; this will prove very useful to readers who wish to go beyond the book itself. Although this monograph was not written as a textbook for class room use, nevertheless, it appears to the reviewer to serve the purpose very well in a first course if supplemented by lectures, examples and laboratory work. The clarity of the text on fundamentals makes it most desirable for the use of beginning students whose mathematical equipment is equivalent to the first year, or better, the first and second years of college mathematics.

Few errors or misprints were observed. On page six and seven the letter d represents the number of deaths in one year among an initial group of l persons exposed—those who die cannot be exposed for a year. On page eighteen in formula (7) the exponent κ should be replaced by k. The general appearance of the monograph, both as to printing and binding, is excellent.

JAMES W. GLOVER

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Annals of Mathematics, volume 28, no. 3. July 1927: "Sextic surfaces with a double septimic curve" by B. C. Wong, 251–262: "Determination of a basis for the integral elements of certain generalized quaternion algebras" by M. D. Darkow, 263–270; "Expansions in Bessel functions" by M. H. Stone, 271–290; "On the Green's function for systems of differential equations" by W. M. Whyburn, 291–300; "On Lamé families of surfaces" by C. E. Weatherburn, 301–308; "Derivation of the Fredholm theory from a differential equation of infinite order" by H. T. Davis, 309–317; "Asymptotic expression for the probability of trials connected in a chain" by E. D. Pepper, 318–326; "On certain indefinite quaternary forms representing all integers" by C. G. Latimer, 327–329; "Applications of algebraic number theory to congruences involving binomial coefficients" by H. S. Vandiver, 330–332; "Ternary quadratic forms and congruences" by L. E. Dickson, 333–341; "Correspondences between algebraic curves" by S. Lefschetz, 342–354; "On the existence and properties of the solution of a certain differential equation of the second order" by T. H. Gronwell, 355–364; "A new definition of almost periodic functions" by N. Wiener, 365–367; "A theory of integers, in relation to the iteration of algebraic functions," by O. E. Glenn, 368–378; "On systems of total differential equations" by J. M. Thomas, 379–385; "On positive solutions of a system of linear equations" by L. L. Dines, 386–392; "On completely signed sets of functions" by L. L. Dines, 393–395; "Concerning continuous curves and correspondences" by W. L. Ayres, 396–418.

Messenger of Mathematics, volume LVI, no. 12, April 1927: "A modification of Glaisher's proof of Stirling's Theorem" by R. E. Moritz, 181–184.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 8, August 1927: "Contact transformations of three-space which convert a system of paths into a system of paths" by Jesse Douglas, 605–606; "Motion and collineations in general space" by M. S. Knebelman, 607–610; "Felix Klein and the history of modern mathematics" by G. A. Miller, 611–613; "The residual set of a complex on a manifold and related questions" by S. Lefschetz, 614–622. No. 9, September 1927: "Remark on a theorem of R. L. Moore" by B. Knaster and C. Kuratowski, 647–649; "Concerning the open subsets of a plane continuous curve" by Gordon T. Whyburn, 650–656; "On the functional independence of ratios of theta functions" by S. Lefschetz, 657–658.

Transactions of the American Mathematical Society, volume 29, no. 3, July 1927: "The contact of a cubic surface with an analytic surface" by E. P. Lane, 471–480; "Dynamical space-times which contain a conformal euclidean 3-space" by H. P. Robertson, 481–496; "Reduction of the ordinary linear differential equation of the nth order whose coefficients are certain polymonials in a parameter to a system of n first-order equations which are linear in the parameter" by C. E. Wilder, 497–506; "On the 'third axiom of metric space'" by C. W. Niemptzki, 507–513; "Implicit functions and differential equations in general analysis" by L. M. Graves, 514–552; "Concerning acyclic continuous curves" by H. M. Gehman, 553-568; "On a general formula in the theory of Tchebycheff polynomials and its applications" by J. A. Shohat, 569–583; "A factorization theory for functions $\sum_{i=1}^{n} a_i e^{\alpha_i x}$" by J. F. Ritt, 584–596;

"Arithmetic of logic" by E. T. Bell, 597–611; "Functionals of r-dimensional manifolds admitting continuous groups of point transformations" by A. D. Michal 612–646.

Isis, volume 9, no. 30, June 1927: "Le problème logique de la définition des nombres irrationnels" by L. Rosenfeld, 345–358.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3288. Proposed by Nathan Altshiller-Court, University of Oklahoma.

The variable line on which two given circles determine chords of equal length envelops a parabola. The midpoint of the line of centers of the circles is the focus of the parabola, and its tangent at the vertex coincides with the radical axis of the two given circles.

3289. Proposed by D. M. Yost, California Institute of Technology.

Evaluate the definite integral $\int_0^\infty e^{-ax}x^{-x}dx$ which is to be regarded as a function of a. If the solution is obtained as a power series in a, discuss the convergence of the series and determine if it is uniformly convergent for all finite real values of a.

3290. Proposed by J. V. Uspensky, Carleton College.

Every positive root of the equation $\tan x = x$ can be expressed as follows: $x=(p+\frac{1}{2})\pi-\theta$, where p is an arbitrary integer $\geqq 0$ and

$$\theta=C_0\xi+C_1\xi^3+C_2\xi^5+\cdots;\quad \xi=1/(p+\tfrac{1}{2})\pi.$$

The coefficients $C_0, C_1, C_2, \cdots$ are positive rational numbers. Show that for n very large

$$C_n=12^{-1/6}\Gamma(1/3)\pi^{-2/3}\cdot(\pi/2)^{2n+1}(2n+1)^{-4/3}(1+\omega_n)$$

where $\omega_n\to 0$ when $n\to\infty$, so that the above series remains convergent even for $p=0$.

3291. Proposed by C. N. Mills, Normal, Illinois.

Given a circle of radius R; AB the side of an inscribed octagon; ACE three consecutive points of the inscribed hexagon; AD the side of an inscribed square. BD intersects CE at O. With O as a center and the chord BC as a radius, describe a circle which intersects AD at P. Show that AP is the side of the inscribed heptagon, with an error of $.000009R$.

3292. Proposed by R. E. Gaines, University of Richmond.

A chord of a given conic is of constant length; find the locus of its pole.

3293. Proposed by R. E. Gaines, University of Richmond.

From the foci F and F' of a conic, lines FP and $F'P'$ are drawn parallel to each other and cutting the conic in P and P'; find the envelope of PP'.

3294. Proposed by A. A. Bennett, Brown University.

Suppose a gambling house establishes an artificial "horse race" with six counters numbered from 1 to 6 representing six competing horses. The banker throws a single die repeatedly, and at each throw one horse is advanced one step from the starting line toward the finishing line. If the die turns up a 1, the counter marked "1" is advanced one step, the other counters being left alone. If 2 turns up, the counter marked "2" is advanced one step, and so on. The race is not an even one, however, since the several horses are given different handicaps. Thus the counter marked "1" has but five steps to go to win the race, while the counter marked "2" has six steps to go, the next, seven, and so on, the counter marked "6" having ten steps to go. A single race terminates as soon as any one horse completes his course. Limited bets may be placed by any player on any horse, and successive races occur unendingly. At the close of each race the banker pays all winning bets. Assume that the banker's profits are derived wholly from general admission fees, determine what odds the banker should offer on each horse, that is, how many dollars should he return to a successful better for each dollar placed on the winning horse, supposing that in the long run the banker may come out approximately even on the bets?

SOLUTIONS

3200 [3188; 1926, 338]. Proposed by B. F. Finkel, Drury College.

Find the equation of the curve whose radius of curvature at any point of the curve is n times the radius vector to the same point.

Solution by P. J. Federico, Washington, D. C.

If the equation of a curve is in polar coordinates, its radius of curvature at any point can be written $R=\rho(d\rho/du)$ where u is the perpendicular distance from the pole to the tangent at the point. (See. Williamson, *Differential Calculus* (1912), page 296.) From the given condition, $R=n\rho$, it follows that $(d\rho/du)=n$, and hence $\rho=nu+c$.

The expression for u is

$$u=\rho^2(d\theta/ds),\quad ds^2=d\rho^2+\rho^2d\theta^2. \tag{1}$$

(Ibid., pages 223, 224). These, when substituted in the equation $\rho=n\,u+c$, give

$$(\rho-c)\rho^{-1}[n^2\rho^2-(\rho-c)^2]^{-1/2}d\rho=d\theta. \tag{2}$$

The integral of (2) is the equation of the required curve in polar coordinates; (2) can always be integrated (using formulas 160 or 161 and 182 or 183 in B. O. Peirce's *Short Table of Integrals*), but the result is, in general, too cumbersome to determine the nature of the curve readily. It will be noticed that the curve is real for all real values of n except $n=0$.

The special case when $c=0$ gives

$$d\rho/\rho=(n^2-1)^{1/2}d\theta;\ \rho=ke^{\mu},\quad \text{where } \mu=(n^2-1)^{1/2}\cdot\theta,$$

which is the partial solution discussed on page 274 of this Monthly for May, 1927.

3220 [1926, 480]. Proposed by W. L. Ayres, University of Pennsylvania.

Find the value of

$$2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{2!}2^{n-4}-\frac{(n-3)(n-4)(n-5)}{3!}2^{n-6}+\cdots$$
$$+(-1)^{r+1}\frac{(n-r+1)(n-r)\cdots(n-2r+3)}{(r-1)!}2^{n-2r+2}+\cdots,$$

where the number of terms is the greatest integer in $(n+2)/2$ and n is a non-negative integer.

Solution by Harry Langman, Brooklyn, N. Y.

The sum may be written

$$A=\sum_{r=0}^{s}(-1)^r\frac{(n-r)(n-r-1)\cdots(n-2r+1)}{r!}2^{n-2r}=2^n\sum_{r=0}^{s}(-1)^rC_r^{n-r}\cdot\frac{1}{4^r}=2^n\sum_{r=0}^{s}a_r,$$

where s is the greatest integer in $n/2$.

Now a_r = coefficient of x^r in $[1-(x/4)]^{n-r}$ = coefficient of x^n in $[x-(x^2/4)]^{n-r}$. Hence $A = 2^n$ coefficient of x^n in

$$\sum_{r=0}^{s} [x - (x^2/4)]^{n-r}.$$

We require then the coefficient of x^n in

$$\Big[[x - (x^2/4)]^{n-s} - [x - (x^2/4)]^{n+1} \Big](1 - x/2)^{-2}.$$

The term $[x-(x^2/4)]^{n+1}$ will not affect the result. Hence we require but the coefficient of x^s in $(1-x/2)^{-2}$ $(1-x/4)^{n-s}$ or

$$\Big(\sum_{a=0}^{\infty} (a+1)x^a/2^a \Big)\Big(\sum_{b=0}^{n-s} (-1)^b C_b^{\,n-s} x^b/4^b \Big).$$

This is evidently

$$\sum_{b=0}^{s} (-1)^b C_b^{\,n-s} \cdot \frac{1}{4^b} \cdot \frac{s-b+1}{2^{s-b}}$$

$$= \frac{s+1}{2^s} \sum_{b=0}^{s} (-1)^b \frac{(n-s)!}{b!(n-s-b)!} \cdot \frac{1}{2^b} + \frac{n-s}{2^{s+1}} \sum_{b=1}^{s} (-1)^{b-1} \frac{(n-s-1)!}{(b-1)!(n-s-b)!} \cdot \frac{1}{2^{b-1}}.$$

If $n = 2s$, this sum becomes

$$\frac{s+1}{2^s}\Big(1 - \frac{1}{2}\Big)^s + \frac{s}{2^{s+1}}\Big(1 - \frac{1}{2}\Big)^{s-1} = \frac{n+1}{2^n}.$$

If $n = 2s+1$, we have

$$\frac{s+1}{2^s}\Big[\Big(1 - \frac{1}{2}\Big)^{s+1} + (-1)^{s+2} \cdot \frac{1}{2^{s+1}}\Big] + \frac{s+1}{2^{s+1}}\Big[\Big(1 - \frac{1}{2}\Big)^s + (-1)^{s+1} \cdot \frac{1}{2^s}\Big]$$

$$= \frac{2s+2}{2^{2s+1}} = \frac{n+1}{2^n}.$$

Hence in either case $A = n+1$.

Also solved by LEONARD CORLITZ and the PROPOSER.

NOTE BY OTTO DUNKEL, Washington University

If we denote the series of the problem by $P_2(n+1)$ it will be found by setting $n = 0, 1, 2, 3$ that $P_2(1) = 1, P_2(2) = 2, P_2(3) = 3, P_2(4) = 4$. This suggests an examination of the second difference. Forming the difference $P_2(n+2) - 2P_2(n+1)$, we easily verify that this difference is equal to $-P_2(n)$. Therefore the second difference is zero for all values of n and hence $P_2(n+1) = An + B$. The particular values above give $A = B = 1$. The value of the series of the problem is then $n+1$.

This series is a special case of a previous result which may be stated as follows:

In n tosses of a single coin the number of ways of tossing at first r or more consecutive heads and in the subsequent tosses no set of a tail followed by as many as r consecutive heads is

$$P_2(n) = \sum_{k=1}^{k=i} (-1)^{k+1}{}_{n-kr}C_{k-1}\, 2^{n-k(r+1)+1}, \qquad i = \Big[\frac{n+1}{r+1}\Big],$$

where ${}_mC_j = m!/j!(m-j)!$ *and* $[\alpha]$ *denotes the greatest integer in* α. Here $P_2(n)$ is a function of n and r. See *Solutions of a Probability Difference Equation* by Otto Dunkel in this Monthly, vol. 32 (1925), pp. 354–370. A different derivation is referred to in that paper.

3224[1926, 481]. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Find the locus of the center of gravity of the variable triangle determined by three skew lines in a plane turning about a fixed axis.

SOLUTION BY MICHAEL GOLDBERG, Washington, D.C.

Let the axis of x be taken along the fixed axis. As the plane turns about this axis it will reach a position, π_n defined by its angle t_n with a fixed reference plane, such that the line l_n projects upon π_n

in a line parallel to the axis. Let l_n make an angle θ_n with π_n and cut it in a point with the coordinates (a_n, p_n) in that plane. When the plane has the position defined by t, it cuts l_n in a point (x_n, y_n) such that

$$y_n = p_n \sec(t - t_n), \quad (x_n - a_n)\tan\theta_n = p_n \tan(t - t_n).$$

If we now shift the origin to the point $\{\frac{1}{3}(a_1+a_2+a_3), 0\}$, the locus of the center of gravity is given by the parametric equations

$$3x = k_1 \tan(t - t_1) + k_2 \tan(t - t_2) + k_3 \tan(t - t_3),$$
$$3y = p_1 \sec(t - t_1) + p_2 \sec(t - t_2) + p_3 \sec(t - t_3)$$

where $k_n = p_n/\tan\theta_n$.

Also solved by THEODORE BENNETT.

3225[1926, 481]. Proposed by C. N. Schmall, New York, N. Y.

If $ABCD$ is a cyclic quadrilateral, and P any point in its plane prove that

$$PA^2 \cdot \Delta BCD + PC^2 \cdot \Delta ABD = PB^2 \cdot \Delta ACD + PD^2 \cdot \Delta ABC.$$

SOLUTION BY ROSCOE WOODS, University of Iowa.

Let (x_i, y_i), $i = 1, 2, 3, 4$, be the coordinates of the points A, B, C, D respectively. No three of these points are collinear. The necessary and sufficient condition that the quadrilateral be cyclic is expressed thus by the vanishing of a fourth order determinant:

$$\begin{vmatrix} x_1^2 + y_1^2, & x_1, y_1, 1 \\ x_2^2 + y_2^2, & x_2, y_2, 1 \\ x_3^2 + y_3^2, & x_3, y_3, 1 \\ x_4^2 + y_4^2, & x_4, y_4, 1 \end{vmatrix} = 0.$$

There is no loss of generality if P be taken at the origin. Note that the elements of the first column are OA^2, OB^2, OC^2, OD^2 respectively. Further note that the minor of OA^2 is twice the area of the triangle BCD and so on for the others. Hence the expansion of the above determinant according to the elements of the first column gives $OA^2 \cdot \Delta BCD + OC^2 \cdot \Delta ABD = OB^2 \cdot \Delta ACD + OD^2 \cdot \Delta ABC$ which is the relation sought. One should consult the following references if interested in this problem from a slightly different viewpoint: Charles Smith, *Conic Sections by the Methods of Coordinate Geometry* (Macmillan and Co., London, 1910), pp. 99–100 and J. L. Coolidge, *A Treatise on the Circle and Sphere* (Clarendon Press, Oxford, 1916), p. 138.

Consider now four points P, Q, R, S in the plane of $ABCD$. From the above we have

$$PA^2 \cdot \Delta BCD + PC^2 \cdot \Delta ABD = PB^2 \cdot \Delta ACD + PD^2 \cdot \Delta ABC$$

and three similar relations for Q, R and S. Now the quantities $\Delta BCD, \Delta ABD$, etc., may be eliminated and we have a relation connecting the squares of the distances from four points and the vertices of a cyclic quadrilateral. This relation is

$$\begin{vmatrix} PA^2 & PB^2 & PC^2 & PD^2 \\ QA^2 & QB^2 & QC^2 & QD^2 \\ RA^2 & RB^2 & RC^2 & RD^2 \\ SA^2 & SB^2 & SC^2 & SD^2 \end{vmatrix} = 0.$$

If P coincides with A, Q with B, etc., this relation becomes

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 \\ BA^2 & 0 & BC^2 & BD^2 \\ CA^2 & CB^2 & 0 & CD^2 \\ DA^2 & DB^2 & DC^2 & 0 \end{vmatrix} = 0$$

which gives rise to Ptolemy's Theorem, namely $AB \cdot CD \pm AC \cdot BD \pm AD \cdot BC = 0$.

Evidently the same procedure may be extended to five co-spherical points in a space of three dimensions. If the points be A, B, C, D, E, no four of which are co-planar, we have immediately

$$\begin{vmatrix} 0 & AB^2 & AC & AD^2 & AE^2 \\ BA^2 & 0 & BC^2 & BD^2 & BE^2 \\ CA^2 & CB^2 & 0 & CD^2 & CE^2 \\ DA^2 & DB^2 & DC^2 & 0 & DE^2 \\ EA^2 & EB^2 & EC^2 & ED^2 & 0 \end{vmatrix} = 0.$$

Consult an article by Cayley on the relation between the distances of five points in space in the Messenger of Mathematics, vol. 18 (1889), pp. 100–102.

If the distance between two points $P(x_1, x_2, \cdots, x_n)$, $P(y_1, y_2, \cdots, y_n)$ in a space of n dimensions is given by

$$\overline{P_1P_2}^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2,$$

then this problem may be extended to a space of n dimensions where the $n+2$ points lie on a hypersphere and no $n+1$ of these points are on a hyperplane.

Also solved by MICHAEL GOLDBERG and HARRY LANGMAN.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Miss GWENTHALYN JONES of Chicago has made a gift of $200,000 for the endowment of a professorship in mathematical physics at Princeton University.

Professor J. S. AMES of Johns Hopkins University has been elected chairman of the National Advisory Committee on Aeronautics.

Professor G. H. CRESSE of the University of Arizona is spending his year of sabbatical leave at the University of Göttingen.

Assistant Professor W. L. CRUM of Harvard University has been appointed professor of statistics in the Graduate School of Business, Stanford University.

Dr. B. F. DOSTAL of the University of Michigan has been appointed assistant professor of mathematics at the University of Florida.

Assistant Professor W. V. N. GARRETSON of Rutgers University has been appointed head of department of mathematics at Ouachita College, Arkadelphia, Arkansas.

Assistant Professor JEWEL C. HUGHES has been promoted to an associate professorship of mathematics at the University of Arkansas.

Professor J. J. KNOX, head of the department of mathematics of the Senior High School of South Bend, Indiana, has been appointed professor of mathematics and head of the department of mathematics at Dakota Wesleyan University.

Dr. F. W. KOKOMOOR of the University of Michigan has been appointed assistant professor of mathematics at the University of Florida.

Dr. C. H. LANGFORD of Howard University has been appointed assistant professor of philosophy at the University of Washington.

Mr. C. A. MESSICK has been promoted to an assistant professorship of mathematics at the University of Florida.

Dr. E. B. MILLER of the University of Wisconsin has been appointed professor of mathematics at Illinois College, Jacksonville, Ill.

Associate Professor FRANK R. MORRIS has been promoted to a full professorship of mathematics at Fresno State College.

Professor F. R. MOULTON has been elected a member of the executive board of the National Research Council.

Dr. J. M. THOMAS has been appointed assistant professor of mathematics at the University of Pennsylvania.

Dr. W. J. TRJITZINSKY has been appointed assistant professor of mathematics at the University of Valparaiso, Valparaiso, Indiana.

Assistant Professor L. A. H. WARREN has been promoted to a full professorship of mathematics at the University of Manitoba.

Dr. LOUIS WEISNER has been appointed assistant professor of mathematics at Hunter College.

Dr. GORDON WHYBURN has been promoted to a full professorship of mathematics at the University of Texas.

The following appointments to instructorships in mathematics are announced:

Johns Hopkins University, Dr. LEONARD M. BLUMENTHAL;
University of Florida, Mr. A. M. CRAIG;
University of Texas, Mr. John H. SIMESTER.

Dr. GEORGE ANDREW HILL, senior astronomer at the United States Naval Observatory, died August 29, 1927, aged 69 years.

Professor H. E. RUSSELL of the University of Denver, since 1918 a member of the Association, died on May 31, 1927, as the result of an automobile accident.

Dr. H. D. THOMPSON, for more than thirty years professor of mathematics at Princeton University, has died, aged 63 years.

Dr. ANNA LAVINIA VAN BENSCHOTEN, professor of mathematics and head of the department of mathematics at Wells College from 1901 to 1920, died at Whittier, California, September 18, 1927.

TWO NOTABLE GIFTS TO THE ASSOCIATION

I. THE CARUS MATHEMATICAL MONOGRAPHS.

The entire expense for publishing and distributing these MONOGRAPHS is provided by MRS. MARY HEGELER CARUS as a gift to the ASSOCIATION. The sale of these books at cost to its members by the ASSOCIATION is thus made possible, and the receipts from such sales are used to build up an endowment fund of the ASSOCIATION to be known as the CARUS PUBLICATION FUND. Hence, when a member buys a CARUS MONOGRAPH he not only gets full value at minimum cost but he also contributes to a fund which will ultimately be of the utmost value to the ASSOCIATION. Can any member show good reason for not rendering this service to the ASSOCIATION? The first and second and third Monographs are still available to members at the cost price. The fourth is in preparation.

II. THE RHIND MATHEMATICAL PAPYRUS.

CHANCELLOR ARNOLD BUFFUM CHACE, of Brown University, who has repeatedly shown his vital interest in the Association by cash contributions to its depleted budget, has now made a notable gift which was fully explained in the September, 1926 issue of the MONTHLY. He has done the ASSOCIATION signal honor by publishing at great expense his RHIND MATHEMATICAL PAPYRUS under its auspices. The entire receipts from the sale of this work will be devoted to an endowment fund of the ASSOCIATION to be known as the ARNOLD BUFFUM CHACE FUND. Individuals and institutions not now members of the ASSOCIATION may secure the special rate to members by making application for membership before the sale begins, early in December.

A more detailed description of this great work is to be found in this November issue of the MONTHLY.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Annual Meeting, Nashville, Tenn., December 29, 30, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 14.

LOUISIANA, MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7, and Georgetown University, December 3.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 6.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November 26.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Austin or Houston, February, 1928.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE FOURTH ANNUAL MEETING OF THE SOUTHEASTERN SECTION

The fourth annual meeting of the Southeastern Section of the Mathematical Association of America was held at Emory University, March 19–20, 1926. There were seventy present, including the following twenty members: E. A. Bailey, D. F. Barrow, L. M. Blumenthal, F. Cumming, L. E. Dickson, M. D. Earle, Floyd Field, R. W. Frary, H. K. Fulmer, Ruby Hightower, J. B. Jackson, A. B. Morton, Frank Ordway, W. P. Ott, W. W. Rankin, Caroline Reaves, D. Rumble, D. M. Smith, R. P. Stephens, Fredrick Wood.

Professor W. P. Ott, vice-chairman, presided on account of the death of the chairman, Professor M. T. Peed. A special dinner was given at the Henry Grady Hotel in honor of Professor L. E. Dickson of the University of Chicago. Professor A. B. Morton was Toastmaster. The following officers were elected: Chairman, W. P. Ott, University of Alabama; Vice Chairman, J. B. Coleman, University of South Carolina; Secretary-Treasurer, W. W. Rankin, Agnes Scott College. The Chairman appointed the following members to act on the Program Committee: W. W. Rankin, chairman, D. F. Barrow, J. W. Lasley, Frank Ordway.

Resolutions were prepared and read by Professors R. P. Stephens and J. F. Messick in memory of Professors S. M. Barton, University of the South and M. T. Peed, Emory University, who had died since the last meeting.

The following program was presented, after which a delightful luncheon was served to the delegates by Emory University.

1. "A chapter in the theory of numbers from the standpoint of quaternions." by Professor L. E. Dickson, University of Chicago. (By invitation.)

2. "Generalizations of certain types of boundary value problems of differential equations," by Professor W. W. Elliott, Duke University.

3. "Transfinite cardinal numbers," by Ruth Linebery, Agnes Scott College. (By invitation.)

4. "New results on algebras and their arithmetics," by Professor L. E. Dickson, University of Chicago.

5. "Applications of mathematics to engineering," by Professor J. B. Peebles, Emory University. (By invitation.)

6. Committee Report: "Closer correlation of college and high school mathematics," by Professor W. W. Rankin, Chairman.

At the special dinner, informal talks were made by Professors Dickson, Field, and Stephens.

W. W. Rankin, *Secretary*